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So-Chin Chen
Mei-Chi Shaw

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Contents

Preface

This book is intended as both an introductory text and a reference book to those interested in studying several complex variables in the context of partial differential equations. The prerequisite is familiarity with real analysis and one complex variable. Some knowledge in distribution theory and elliptic partial differential equations will also be helpful.

The Cauchy-Riemann operator is of fundamental importance in complex analysis. When restricted to the boundary, it naturally induces the tangential Cauchy-Riemann operator in several complex variables. In the last few decades, tremendous progress has been made in the study of these two operators which has greatly influenced the development of partial differential equations as well as several complex variables. The purpose of this book is to provide an updated account of recent developments for these equations and related operators.

The main feature of each of these operators is that it is no longer elliptic, either as a boundary value problem or as an operator by itself. We first use L^2 theory to study the Cauchy-Riemann equations and the $\bar{\partial}$-Neumann problem with *a priori* estimates and harmonic analysis. Pseudodifferential operators are used only when we study the tangential Cauchy-Riemann complex and its Laplacian. The integral kernel method is also employed to obtain explicit formulas for solutions with estimates in Hölder and L^p spaces. These techniques are applied to obtain Hodge type decomposition theorems for the Cauchy-Riemann complex and for the tangential Cauchy-Riemann complex. Another topic extensively treated in this book is the realization of almost complex manifolds and the embeddability of abstract CR structures, both locally and globally.

In the first three chapters, we introduce some background material in complex analysis. This includes the Cauchy integral formula in one variable with its applications, domains of holomorphy and pseudoconvexity in several variables.

Chapters 4-6 are devoted to the solvability and regularity of the Cauchy-Riemann equation using Hilbert space techniques. In Chapter 4 we derive L^2 existence theorems for $\bar{\partial}$ on pseudoconvex domains using estimates which involve densities depending on a parameter. The existence of the $\bar{\partial}$-Neumann operator is also obtained. On strongly pseudoconvex domains, subelliptic 1/2-estimates are deduced for the $\bar{\partial}$-Neumann operator and the boundary regularity in Sobolev spaces is studied in Chapter 5. In Chapter 6, the vector field technique is introduced to study the global regularity of the $\bar{\partial}$-Neumann problem on weakly pseudoconvex domains. Boundary regularity of the Bergman projection and biholomorphic mappings are also investigated.

The second half of the book is intended as a self-contained introduction to the tangential Cauchy-Riemann equation. In Chapter 7, the tangential Cauchy-Riemann complex is defined on a CR manifold. The nonsolvable Lewy operator is then shown to arise from the tangential Cauchy-Riemann operator associated with a strongly pseudoconvex domain. In Chapters 8 and 9, we give a detailed account of the L^2 theory of $\bar{\partial}_b$. In Chapter 8, using pseudodifferential operators, we establish the subelliptic estimate for $\Box_b$ which is essential to the existence and regularity theorems for $\overline{\partial}_b$ on compact strongly pseudoconvex CR manifolds. Hypoellipticity of sum of squares of vector fields of finite type is also proved. Chapter 9 is devoted to L^2 existence theorems and the closed-range property for $\overline{\partial}_b$ on the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^n$.

In Chapters 10 and 11, integral representations of solutions are the main theme. This includes the construction of an explicit fundamental solution for $\Box_b$ on the Heisenberg group, and necessary and sufficient conditions for the local solvability of Lewy operator. Integral kernels for $\bar{\partial}$ and $\bar{\partial}_b$ are derived with estimates on convex domains and their boundaries. We also derive a homotopy formula for $\overline{\partial}_b$ over an open subset on a strongly pseudoconvex hypersurface with L^p estimates and interior regularity. In Chapter 12, embeddability of abstract CR structures is addressed. This includes both local and global embeddability results and counterexamples for strongly pseudoconvex CR structures.

Many topics are omitted in order to keep the book a reasonable length. In particular, finite type condition and compactness for the $\overline{\partial}$-Neumann problem, analytic hypoellipticity for subelliptic operators, function theory on Stein manifolds or nonsmooth domains, the asymptotic expansion of the Bergman kernel, and the complex Monge-Ampère equation are not dealt with here. Pseudodifferential operators and integral kernel methods are only used in the simplest setting. Fortunately, many excellent textbooks and references are available on these topics. The books *The Neumann Problem for the Cauchy-Riemann Complex* by G. B. Folland and J. J. Kohn and *An Introduction to Complex Analysis in Several Variables* by L. Hörmander have been inspirations to both authors. Several excellent textbooks are also available on integral kernel methods. Numerous references are given in the notes after each chapter. These are by no means complete, in light of the recent activity, but are intended to direct interested readers.

It gives us great pleasure to thank our thesis advisor, Professor Joseph J. Kohn, for introducing us to this beautiful subject. His influence is evident throughout the book. We are indebted to David Barrett, Harold P. Boas, Dan Coman, Dariush Ehsani, Phillip Harrington, Alex Himonas, Michael Range, Nancy Stanton, Sophia Vassiliadou and Deyun Wu, who have read part of the manuscript and provided many valuable suggestions. Annette Pilkington has given us editorial assistance. Professor Shing-Tung Yau kindly invited us to publish this book at the AMS-International Press. We would like to thank them all.

We would also like to thank our home institutes for their support. Financial assistance from the National Science Foundation of the United States, the National Science Council of the Republic of China in Taiwan, and the University of Notre Dame are gratefully acknowledged. Finally, we would like to express our deepest appreciation for our spouses, Jen-Fen Y. Chen and Hsueh-Chia Chang, over the past few years. Without their patience and understanding, the book would not have been finished.

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APPENDIX

A. Sobolev Spaces

We include a short summary of the basic properties of the Sobolev spaces for the convenience of the reader. Our goal is to give precise definitions and statements of all theorems or lemmas about the Sobolev spaces which have been used in this book. Since most of the results are well-known and due to the vast amount of literature on this subject, we will provide very few proofs.

Let $f \in L^1(\mathbb{R}^N)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \tag{1.1}$$

where $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$. The estimate

$$\| \hat{f} \|_\infty \leq \| f \|_{L^1}$$

is clear from the definition. We now list some basic properties of the Fourier transform whose proofs are left to the reader or can be found in any standard text. For instance, see Stein-Weiss [StWe 1].

Theorem A.1 (Riemann-Lebesgue). *Suppose that $f \in L^1(\mathbb{R}^N)$, then $\hat{f}(\xi) \in C_0$, where C_0 denotes the space of continuous functions on $\mathbb{R}^N$ that vanish at infinity.*

Theorem A.2 (Fourier Inversion). *Suppose that $f \in L^1(\mathbb{R}^N)$ and that $\hat{f}(\xi) \in L^1(\mathbb{R}^N)$. Then*

$$f(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi, \quad a.e.$$

In other words, $f(x)$ can be redefined on a Lebesgue measure zero set so that $f(x) \in C_0$.

Theorem A.3 (Uniqueness). *If $f \in L^1(\mathbb{R}^N)$ and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}^N$, then $f(x) = 0$ almost everywhere.*

Denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^N$, i.e., $\mathcal{S}$ consists of all smooth functions f on $\mathbb{R}^N$ with

$$\sup_{\mathbb{R}^N} |x^\beta D^\alpha f(x)| < \infty,$$

for all multiindices α, β, where $\alpha = (\alpha_1, \cdots, \alpha_N)$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_N}^{\alpha_N}$, each α_i is a nonnegative integer. Obviously, any smooth function with compact support belongs to $\mathcal{S}$ and we have the following formulas:

$$\begin{aligned} (\widehat{D^\alpha f})\ (\xi) &= (i\xi)^\alpha \hat{f}(\xi). \\ D^\alpha \hat{f}(\xi) &= ((\widehat{-ix)^\alpha} f)(\xi). \end{aligned} \tag{1.2}$$

Theorem A.4. *The Fourier transform is an isomorphism from $\mathcal{S}$ onto itself.*

Since $L^2(\mathbb{R}^N) \not\subseteq L^1(\mathbb{R}^N)$, the Fourier transform defined by (1.1) in general cannot be applied to L^2 functions directly. Using the following fundamental theorem of the Fourier transform, one can extend the definition to L^2 functions easily:

Theorem A.5 (Plancherel's Theorem). *The Fourier transform can be extended to be an automorphism of $L^2(\mathbb{R}^N)$ with*

$$\| \hat{f} \|^2 = (2\pi)^N \| f \|^2 \quad \text{for all } f \in L^2(\mathbb{R}^N). \tag{1.3}$$

Equation (1.3) is called the Parseval's identity.

We collect a few results about the Sobolev spaces. For a detailed treatment of the Sobolev spaces $W^s(\Omega)$ for any real s, we refer the reader to Chapter 1 in Lions-Magenes [LiMa 1] for smooth domains or to Grisvard [Gri 1] for nonsmooth domains.

We first define the Sobolev spaces in $\mathbb{R}^N$. Let

$$p(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$$

be a differential operator of order m with constant coefficients. Then, by (1.2), it is easy to see that for any $f \in \mathcal{S}$,

$$(\widehat{p(D)f})(\xi) = p(i\xi)\hat{f}(\xi). \tag{1.4}$$

Here, the polynomial $p(i\xi)$ is obtained by replacing the operator D in $p(D)$ by $i\xi$.

For any $s \in \mathbb{R}$, we define $\Lambda^s : \mathcal{S} \to \mathcal{S}$ by

$$\Lambda^s u(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi}(1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\, d\xi. \tag{1.5}$$

Set $\sigma(\Lambda^s) = (1 + |\xi|^2)^{s/2}$. $\sigma(\Lambda^s)$ is called the symbol of Λ^s. Define the scalar product $(u, v)_s$ on $\mathcal{S} \times \mathcal{S}$ by

$$(u, v)_s = (\Lambda^s u, \Lambda^s v)$$

and the norm

$$\| u \|_s = \sqrt{(u, u)_s} \quad \text{for } u \in \mathcal{S}.$$

The Sobolev space $H^s(\mathbb{R}^N)$ is the completion of $\mathcal{S}$ under the norm defined above. In particular, $L^2(\mathbb{R}^N) = H^0(\mathbb{R}^N)$. The Sobolev norms $\| \ \|_{H^s(\mathbb{R}^N)}$ for any $u \in C_0^\infty(\mathbb{R}^N)$ is given by

$$\| u \|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi. \tag{1.6}$$

Next, we define the Sobolev spaces for domains in $\mathbb{R}^N$. Let $\Omega \subset\subset \mathbb{R}^N$ be a domain with C^k boundary, $k = 1, 2, \cdots$. By this we mean that there exists a real-valued C^k function ρ defined in $\mathbb{R}^N$ such that $\Omega = \{z \in \mathbb{R}^N | \rho(z) < 0\}$ and $|\nabla\rho| \ne 0$ on $b\Omega$.

The implicit function theorem shows that locally, $b\Omega$ can always be expressed as a graph of a C^k function. If the boundary can be expressed locally as the graph of a Lipschitz function, then it is called a Lipschitz domain or a domain with Lipschitz boundary.

For any domain Ω in $\mathbb{R}^N$, let $H^s(\Omega)$, $s \geq 0$, be defined as the space of the restriction of all functions $u \in H^s(\mathbb{R}^N)$ to Ω. We define the norm of $H^s(\Omega)$ by

$$\| u \|_{H^s(\Omega)} = \inf_{\substack{\mathcal{U} \in H^s(\mathbb{R}^N) \\ \mathcal{U}|_\Omega = u}} \| \mathcal{U} \|_{s(\mathbb{R}^N)} . \tag{1.7}$$

When s is a positive integer, there is another way to define the Sobolev spaces by weak derivatives. For any domain $\Omega \subset \mathbb{R}^N$, we define $W^s(\Omega)$ to be the space of all the distributions u in $L^2(\Omega)$ such that

$$D^\alpha u \in L^2(\Omega), \quad |\alpha| \leq s,$$

where α is a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We define the norm $\| \ \|_{W^s(\Omega)}$ by

$$\| u \|^2_{W^s(\Omega)} = \sum_{|\alpha| \leq s} \| D^\alpha u \|^2_{(\Omega)} < \infty. \tag{1.8}$$

The space $C^\infty(\overline{\Omega})$ denotes the space of functions which are restrictions of functions in $C^\infty(\mathbb{R}^N)$ to $\overline{\Omega}$. If Ω is a bounded Lipschitz domain, then $C^\infty(\overline{\Omega})$ is dense in $W^s(\Omega)$ in the $W^s(\Omega)$ norm (see Theorem 1.4.2.1 in Grisvard [Gri 1]). Thus $W^s(\Omega)$ can also be defined as the completion of the functions of $C^\infty(\overline{\Omega})$ under the norm (1.8) when Ω has Lipschitz boundary.

When $\Omega = \mathbb{R}^N$, we have $H^s(\mathbb{R}^N) = W^s(\mathbb{R}^N)$ for any positive integer s. This follows from Plancherel's theorem and the inequality

$$\frac{1}{C} \sum_{|\alpha| \leq s} |\xi^\alpha|^2 \leq (1 + |\xi|^2)^s \leq C \sum_{|\alpha| \leq s} |\xi^s|^2,$$

where $C > 0$.

Obviously for any bounded domain Ω, we have $H^s(\Omega) \subseteq W^s(\Omega)$ for any Ω. If $b\Omega$ is Lipschitz, the following theorem shows that the two spaces are equal:

Theorem A.6 (Extension Theorem). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. For any positive integer s, there exists a continuous linear operator P_s from $W^s(\Omega)$ into $W^s(\mathbb{R}^N)$ such that*

$$P_s u|_\Omega = u.$$

The extension operator P_s can be chosen to be independent of s. In particular, we have

$$W^s(\Omega) = H^s(\Omega).$$

For a proof of Theorem A.6, see Chapter 6 in Stein [Ste 2] or Grisvard [Gri 1]. Thus when s is a positive integer and Ω is bounded Lipschitz, the Sobolev spaces will be denoted by $W^s(\Omega)$ with norm $\| \ \|_{s(\Omega)}$, or simply $\| \ \|_s$

Theorem A.7 (Sobolev Embedding). *If Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, then there is an embedding*

$$W^k(\Omega) \hookrightarrow C^m(\overline{\Omega}) \quad \textit{for any interger } m, \;\; 0 \le m < k - N/2.$$

Theorem A.8 (Rellich Lemma). *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. If $s > t \ge 0$, the inclusion $W^s(\Omega) \hookrightarrow W^t(\Omega)$ is compact.*

Theorem A.9 (Trace Theorem). *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. For $s > 1/2$, the restriction map $f \to f\big|_{b\Omega}$ for any $f \in C^\infty(\overline{\Omega})$ can be extended as a bounded operator from $W^s(\Omega)$ to $W^{s-1/2}(b\Omega)$. For any $f \in W^s(\Omega)$, $f\big|_{b\Omega} \in W^{s-1/2}(b\Omega)$ and there exists a constant C_s independent of f such that*

$$\| f \|_{s-\frac{1}{2}(b\Omega)} \le C_s \| f \|_{s(\Omega)} .$$

We remark that in general, the trace theorem does not hold for $s = 1/2$. However, if $f \in W^{1/2}(\Omega)$ and f is harmonic or f satisfies some elliptic equations, then the restriction of f to $b\Omega$ is in L^2 (c.f. Lemma 5.2.3).

Let Ω be a bounded domain in $\mathbb{R}^N$. We introduce other Sobolev spaces. Let $W_0^s(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under $W^s(\Omega)$ norm. When $s = 0$, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $W_0^0(\Omega) = W^0(\Omega) = L^2(\Omega)$. If $s \le 1/2$, we also have $C_0^\infty(\Omega)$ is dense in $W^s(\Omega)$. Thus

$$W^s(\Omega) = W_0^s(\Omega), \quad s \le \frac{1}{2}.$$

This implies that the trace theorem does not hold for $s \le 1/2$. When $s > 1/2$, $W_0^s(\Omega) \subsetneq W^s(\Omega)$.

We define $W^{-s}(\Omega)$ to be the dual of $W_0^s(\Omega)$ when $s > 0$ and the norm of $W^{-s}(\Omega)$ is defined by

$$\| f \|_{-s(\Omega)} = \sup \frac{|(f,g)|}{\| g \|_{s(\Omega)}},$$

where the supremum is taken over all functions $g \in C_0^\infty(\Omega)$. We note that the generalized Schwarz inequality for $f \in W^s(\Omega)$, $g \in W^{-s}(\Omega)$,

$$|(f,g)_\Omega| \le \| f \|_{s(\Omega)} \| g \|_{-s(\Omega)}$$

holds only when $s \le 1/2$ for a bounded domain Ω. The proof of these results can be found in Lions-Magenes [LiMa 1] or Grisvard [Gri 1].

The Sobolev spaces can also be defined for functions or forms on manifolds. Let M be a compact Riemannian manifold of real dimension N. Choose a finite number of coordinate neighborhood systems $\{(U_i, \varphi_i)\}_{i=1}^m$, where

$$\varphi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^N$$

is a homeomorphism from U_i onto an open subset V_i contained in $\mathbb{R}^N$. For each i, $1 \le i \le m$, let $\{\eta_j^i\}_{j=1}^N$ be an orthonormal basis for $\mathbb{C}T^*(M)$ on U_i, and let $\{\zeta_i\}_{i=1}^m$

be a partition of unity subordinate to $\{U_i\}_{i=1}^m$. Thus, locally on each coordinate chart U_i, one may express a smooth r-form ϕ as

$$\phi = \sum_{|I|=r}{}' \phi_I^i \eta_I^i, \tag{1.9}$$

where $I = (i_1, \cdots, i_r)$ and $\eta_I^i = \eta_{i_1}^i \wedge \cdots \wedge \eta_{i_r}^i$. Then, we define the Sobolev s norm of $\phi \in \mathcal{E}^r(M)$, for $s \in \mathbb{R}$, by

$$\| \phi \|_s^2 = \sum_{i=1}^m \sum_{|I|=r}{}' \| (\zeta_i \phi_I^i) \circ \varphi_i^{-1} \|_s^2 . \tag{1.10}$$

Denote by $W_r^s(M)$ the completion of $\mathcal{E}^r(M)$ under the norm $\| \cdot \|_s$. The definition of $W_r^s(M)$ is highly nonintrinsic. Obviously, it depends on the choice of the coordinate neighborhood systems, the partition of unity and the local orthonormal basis $\{\eta_j^i\}$. However, it is easily seen that different choices of these candidates will come up with an equivalent norm. Therefore, $W_r^s(M)$ is a well-defined topological vector space. If M is a complex manifold of dimension n and Ω is a relatively compact subset in M, the space $W_{(p,q)}^s(\Omega)$, $0 \le p, q \le n$ and $s \in \mathbb{R}$, are defined similarly. The Sobolev embedding theorem and the Rellich lemma also hold for manifolds.

B. Interpolation Theorems and some Inequalities

There is yet another way to define the Sobolev spaces $W^s(\Omega)$ when s is not an integer and $s > 0$. Let k_1 and k_2 be two nonnegative integers and $k_1 > k_2$. On any domain Ω in $\mathbb{R}^N$, we have $W^{k_1}(\Omega) \subset W^{k_2}(\Omega)$. The space $W^s(\Omega)$ for $k_2 < s < k_1$ can be defined by interpolation theory. We shall describe the procedure in detail for the interpolation between W^1 and L^2 (i.e., when $k_1 = 1$ and $k_2 = 0$).

For each $v \in W^1(\Omega)$ and $u \in W^1(\Omega)$,

$$(u, v)_1 \;=\; (u, v) + \sum_{i=1}^N (D_i u, D_i v),$$

where $D_i = \partial/\partial x_i$. Let $D(\mathcal{L})$ denote the set of all functions u such that the linear map

$$v \longrightarrow (u, v)_1, \quad v \in W^1(\Omega)$$

is continuous in $L^2(\Omega)$. From the Hahn-Banach theorem and the Riesz representation theorem, there exists $\mathcal{L}u \in L^2(\Omega)$ such that

$$(u, v)_1 = (\mathcal{L}u, v), \quad v \in W^1(\Omega). \tag{2.1}$$

If $u \in C_0^\infty(\Omega)$, then $u \in D(\mathcal{L})$ and $\mathcal{L}u = (-\triangle + 1)u$. It is easy to see that $\mathcal{L}$ is a densely defined, unbounded self-adjoint operator and $\mathcal{L}$ is strictly positive since

$$(\mathcal{L}u, u) = \|u\|_1^2 \ge \|u\|^2.$$

Using the spectral theory of positive self-adjoint operators (see e.g. Riesz-Nagy [RiNa 1]), we can define $\mathcal{L}^\theta$ of $\mathcal{L}$ for $\theta \in \mathbb{R}$. Let

$$\Lambda = \mathcal{L}^{1/2}.$$

Then Λ is self-adjoint and positive in $L^2(\Omega)$ with domain W^1. From (2.1), we have

$$(u, v)_1 = (\Lambda u, \Lambda v), \quad \text{for every } u,\ v \in W^1(\Omega).$$

Definition B.1. *Let $W^\theta(\Omega)$ be the interpolation space between the spaces $W^1(\Omega)$ and $L^2(\Omega)$ defined by*

$$W^\theta(\Omega) \equiv [W^1(\Omega), L^2(\Omega)]_\theta = Dom(\Lambda^{1-\theta}), \quad 0 \leq \theta \leq 1,$$

with norm

$$\|u\| + \|\Lambda^{1-\theta} u\| = \text{the norm of the graph of } \Lambda^{1-\theta},$$

where $Dom(\Lambda^{1-\theta})$ denotes the domain of $\Lambda^{1-\theta}$.

From the definition, we have the following interpolation inequality:

$$\|\Lambda^{1-\theta} u\| \leq \|\Lambda u\|^{1-\theta} \|u\|^\theta \tag{2.2}$$

Thus

$$\|u\|_\theta \leq C \|u\|_1^{1-\theta} \|u\|^\theta. \tag{2.3}$$

The general case for arbitrary integers k_1 and k_2 can be done similarly. Thus, this gives another definition for the Sobolev spaces $W^s(\Omega)$ when s is not an integer. If $b\Omega$ is bounded Lipschitz, this space is the same Sobolev space as the one introduced in Appendix A (see [LiMa 1] for details for the equivalence of these spaces). For a bounded Lipschitz domain, we can use any of the definitions for $W^s(\Omega)$, $s \geq 0$.

The following interpolation inequality holds for general Sobolev spaces:

Theorem B.2 (Interpolation Inequality). *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. For any $\epsilon > 0$, $f \in W^{s_1}(\Omega)$, $s_1 > s > s_2 \geq 0$, we have the following inequality:*

$$\| f \|_s^2 \leq \epsilon \| f \|_{s_1}^2 + C_\epsilon \| f \|_{s_2}^2, \tag{2.4}$$

where C_ϵ is independent of f.

Theorem B.3 (Interpolation Theorem). *Let T be a bounded linear operator from $W^{s_i}(\Omega)$ into $W^{t_i}(\Omega)$, $i = 1,\ 2$, and*

$$s_1 > s_2 \geq -\frac{1}{2},\ t_1 > t_2 \geq -\frac{1}{2},$$

then T is bounded from $[W^{s_1}(\Omega),\ W^{s_2}(\Omega)]_\theta$ into $[W^{t_1}(\Omega), W^{t_2}(\Omega)]_\theta$, $0 \leq \theta \leq 1$.

We warn our reader of the danger of interpolation of spaces if the assumption $s_i \geq -1/2$ and $t_i \geq -1/2$ is dropped! (See [LiMa 1].) Next we discuss the interpolation between L^p spaces and some applications.

Definition B.4. *Let (X, μ) and (Y, ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X, μ) into measurable functions defined on (Y, ν). T is called an operator of type (p, q) if there exists a constant $M > 0$ such that*

$$\| Tf \|_{L^q} \leq M \| f \|_{L^p} \tag{2.5}$$

for all $f \in L^p(X)$.

The least M for which inequality (2.5) holds is called the (p,q)-norm of T. If f is a measurable function on (X, μ), we define its *distribution function* $\lambda_f : (0, \infty) \to [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x \mid |f(x)| > \alpha\}).$$

Definition B.5. *Let (X,μ) and (Y,ν) be two measure spaces and let T be a linear operator from a linear subspace of measurable functions on (X,μ) into measurable functions defined on (Y,ν). T is a linear operator of weak type (p,q), $1 \le p \le \infty$ and $1 \le q < \infty$, if there exists a constant k such that*

$$\lambda(s) \le \left(\frac{k \parallel f \parallel_{L^p}}{s}\right)^q \quad \text{for every } f \in L^p(X),$$

where λ is the distribution function of Tf.

We have the following interpolation theorems:

Theorem B.6 (Riesz-Thorin). *Let (X,μ) and (Y,ν) be two measure spaces and p_0, p_1, q_0, q_1 be numbers in $[1,\infty]$. If T is of type (p_i,q_i) with (p_i,q_i)-norm M_i, $i=0,1$, then T is of type (p_t,q_t) and*

$$\parallel Tf \parallel_{L^{q_t}} \le M_0^{1-t} M_1^t \parallel f \parallel_{L^{p_t}}, \tag{2.6}$$

provided

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.

For proof of this fact, see Theorem 1.3 in Chapter 5 in Stein-Weiss [StWe 1].

Theorem B.7 (Marcinkiewicz). *Let (X,μ) and (Y,ν) be two measure spaces and p_0, p_1, q_0, q_1 be numbers such that $1 \le p_i \le q_i \le \infty$ for $i=0,1$ and $q_0 \ne q_1$. If T is of weak type (p_i,q_i), $i=0,1$, then T is of type (p_t,q_t) provided*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

with $0 < t < 1$.

For a proof of this theorem, see Appendix B in Stein [Ste 3].

Theorem B.8 (Hardy's Inequality). *If $f \in L^p(0,\infty)$, $1 < p \le \infty$ and*

$$Tf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0,$$

then

$$\|Tf\|_{L^p} \le \frac{p}{p-1}\|f\|_{L^p}.$$

Proof. We use a change of variables and Minkowski's inequality for integrals,

$$\begin{aligned}\|Tf(x)\|_{L^p} &= \left\|\int_0^1 f(tx)dt\right\|_p \le \int_0^1 \|f(tx)\|_p dt \\ &= \int_0^1 \|f\|_p \frac{1}{t^{\frac{1}{p}}} dt = \frac{p}{p-1}\|f\|_{L^p}.\end{aligned}$$

Theorem B.9. *Let*

$$Tf(x) = \int_0^\infty K(x,y)f(y)dy, \quad x > 0,$$

where $K(x,y)$ is homogeneous of degree -1, that is, $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$, for $\lambda > 0$. If for each $1 \leq p \leq \infty$,

$$\int \mid K(1,y) \mid y^{-1/p}dy = A_p < \infty,$$

then

$$\|Tf\|_{L^p} \leq A_p\|f\|_{L^p}, \qquad \textit{for every } f \in L^p(0,\infty).$$

In particular, the Hilbert integral defined by

$$Tf(x) = \int_0^\infty \frac{f(y)}{x+y}dy, \quad x > 0,$$

is a bounded operator of type (p,p) for each $1 < p < \infty$.

Proof. Since

$$Tf(x) = \int_0^\infty K(1,y)f(xy)dy,$$

using Minkowski's inequality for integrals, we get

$$\|Tf\|_{L^p} \leq \left(\int \mid K(1,y) \mid y^{-1/p}dy\right) \|f\|_{L^p} = A_p\|f\|_{L^p}.$$

The Hilbert integral is of type (p,p) since, for $1 < p < \infty$, using contour integration, we have

$$\int \frac{y^{-1/p}}{1+y}dy = \frac{\pi}{\sin(\pi/p)}.$$

Theorem B.10. *Let (X,μ) and (Y,ν) be two measure spaces and let $K(x,y)$ be a measurable function on $X \times Y$ such that*

$$\int_X |K(x,y)| \, d\mu \leq C, \quad \textit{for a.e.} \quad y,$$

and

$$\int_Y |K(x,y)| \, d\nu \leq C, \quad \textit{for a.e.} \quad x,$$

where $C > 0$ is a constant. Then, for $1 \leq p \leq \infty$, the operator T defined by

$$Tf(x) = \int_Y K(x,y)f(y) \, d\nu$$

is a bounded linear operator from $L^p(Y,d\nu)$ into $L^p(X,d\mu)$ with

$$\| Tf \|_{L^p(X)} \leq C \| f \|_{L^p(Y)} .$$

For a proof of Theorem B.10, we refer the reader to Theorem 6.18 in Folland [Fol 3].

Theorem B.11. *Let (X, μ) and (Y, ν) be two measure spaces and $1 < q < \infty$. Let $K(x, y)$ be a measurable function on $X \times Y$ such that*

$$\nu\{\ y \in Y \mid K(x,y) > s\} \leq \left(\frac{C}{s}\right)^q, \quad \text{for a.e. } x \in X,$$

and

$$\mu\{\ x \in X \mid K(x,y) > s\} \leq \left(\frac{C}{s}\right)^q, \quad \text{for a.e. } y \in Y,$$

where $C > 0$ is a constant. Then the operator T defined by

$$Tf(x) = \int_Y K(x,y) f(y)\, d\nu$$

is a bounded linear operator from $L^p(Y)$ into $L^r(X)$ provided

$$1 < p < r < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1.$$

T is bounded from $L^1(Y)$ to $L^{q-\epsilon}(X)$ for any $\epsilon > 0$.

The proof of this theorem is based on the Marcinkiewicz Interpolation Theorem B.7. We refer the reader to Theorem 15.3 in Folland-Stein [FoSt 1] or Theorem 6.35 in Folland [Fol 3].

C. Hardy-Littlewood Lemma and its Variations

We first prove the Hardy-Littlewood lemma for bounded Lipschitz domains.

Theorem C.1 (Hardy-Littlewood Lemma). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ denote the distance function from x to the boundary of Ω. If u is a C^1 function in Ω and there exists an $0 < \alpha < 1$ and $C > 0$ such that*

$$|\nabla u(x)| \leq C\ \delta(x)^{-1+\alpha} \text{ for every } x \in \Omega, \tag{3.1}$$

then $u \in C^\alpha(\Omega)$, i.e., there exists some constant C_1 such that

$$|u(x) - u(y)| \leq C_1 |x - y|^\alpha \quad \text{for } x, y \in \Omega.$$

Proof. Since u is C^1 in the interior of Ω, we only need to prove the assertion when x and y are near the boundary. Using a partition of unity, we can assume that u is supported in $U \cap \overline{\Omega}$, where U is a neighborhood of a boundary point $x_0 \in b\Omega$. After a linear change of coordinates, we may assume $x_0 = 0$ and for some $\varepsilon > 0$,

$$U \cap \Omega = \{x = (x', x_N) \mid x_N > \phi(x'), |x'| < \varepsilon, |x_N| < \varepsilon\},$$

where $\phi(0) = 0$ and ϕ is some Lipschitz function with Lipschitz constant M. The distance function $\delta(x)$ is comparable to $x_N - \phi(x')$, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C}\delta(x) \leq x_N - \phi(x') \leq C\delta(x) \quad \text{for } x \in \Omega. \tag{3.2}$$

We set $\tilde{x}' = \theta x' + (1-\theta)y'$ and $\tilde{x}_N = \theta x_N + (1-\theta)y_N$. Let $d = |x-y|$. If $x = (x', x_N)$, $y = (y', y_N) \in \Omega$, then the line segment L defined by $\theta(x', x_N + Md) + (1-\theta)(y', y_N + Md) = (\tilde{x}', \tilde{x}_N + Md)$, $0 \le \theta \le 1$, lies in Ω since

$$\begin{aligned}\theta(x_N + Md) &+ (1-\theta)(y_N + Md)\\ &\ge Md + \theta\phi(x') + (1-\theta)\phi(y')\\ &\ge Md + \theta(\phi(x') - \phi(\tilde{x}')) + (1-\theta)(\phi(y') - \phi(\tilde{x}')) + \phi(\tilde{x}')\\ &\ge \phi(\tilde{x}').\end{aligned}$$

Since u is C^1 in Ω, using the mean value theorem, there exists some $(\tilde{x}', \tilde{x}_N + Md) \in L$ such that

$$| u(x', x_N + Md) - u(y', y_N + Md) | \le | \nabla u(\tilde{x}', \tilde{x}_N + Md) | \, d.$$

From (3.1) and (3.2), it follows that

$$\begin{aligned}| u(x', x_N + Md) - u(y', y_N + Md) | &\le C\delta(\tilde{x}', \tilde{x}_N + Md)^{-1+\alpha} \cdot d\\ &\le \tilde{C}((M+1)d)^{-1+\alpha} \cdot d \le C_M d^\alpha.\end{aligned}$$

Also we have

$$\begin{aligned}&| u(x) - u(x', x_N + Md) |\\ &= \left| \int_0^{Md} \frac{\partial u(x', x_N + t)}{\partial t} dt \right|\\ &\le C \int_0^{Md} \delta(x', x_N + t)^{-1+\alpha} dt \le C \int_0^{Md} (x_N + t - \phi(x'))^{-1+\alpha} dt\\ &\le C \int_0^{Md} t^{-1+\alpha} dt \le C(Md)^\alpha.\end{aligned}$$

Thus for any $x, \ y \in \Omega$,

$$\begin{aligned}| u(x) - u(y) | \le | (u(x) - u(x', x_N + Md) | &+ | u(y', u_N + Md) - u(y) |\\ &+ | u(x', x_N + Md) - u(y', y_N + Md) |\\ &\le C_M d^\alpha.\end{aligned}$$

This proves the theorem.

The following is a variation of the Hardy-Littlewood lemma for Sobolev spaces.

Theorem C.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ be the distance function from x in Ω to the boundary $b\Omega$. If $u \in L^2(\Omega) \cap W^1_{\text{loc}}(\Omega)$ and there exists an $0 < \alpha < 1$ such that*

$$\int_\Omega \delta(x)^{2-2\alpha} \,|\, \nabla u \,|^2 \, dV < \infty, \tag{3.3}$$

then $u \in W^\alpha(\Omega)$. *Furthermore, there exists a constant* C*, depending only on* Ω*, such that*

$$\| u \|^2_{\alpha(\Omega)} \le C \left(\int_\Omega \delta(x)^{2-2\alpha} \mid \nabla u \mid^2 dV + \int_\Omega |u|^2 dV \right).$$

Proof. For $0 < \alpha < 1$, $W^\alpha(\Omega) = [W^1(\Omega), L^2(\Omega)]_{1-\alpha}$. The interpolation norm of a function u in $W^\alpha(\Omega)$ (see Lions-Magenes [LiMa 1]) is comparable to the infimum over all functions

$$f \ : [0,\infty) \to L^2(\Omega) + W^1(\Omega) \quad \text{with } f(0) = u$$

of the norm I_f where I_f is defined to be

$$I_f = \left(\int_0^\infty \|t^{1-\alpha} f(t)\|^2_{W^1(\Omega)} t^{-1} dt \right)^{\frac{1}{2}} + \left(\int_0^\infty \|t^{1-\alpha} f'(t)\|^2_{L^2(\Omega)} t^{-1} dt \right)^{\frac{1}{2}}. \tag{3.4}$$

From (3.3), we have $u \in W^1(\Omega')$ for any $\Omega' \subset\subset \Omega$. Thus we only need to estimate u in a small neighborhood of the boundary. Using a partition of unity and a change of coordinates as in Theorem C.1, we can assume $U \cap \Omega = \{x_N > \phi(x')\}$. Let $\eta \in C_0^\infty(-\varepsilon, \varepsilon)$ such that $0 \le \eta \le 1, \eta \equiv 1$ when $\mid t \mid < \varepsilon/2$. We define

$$f(t) = u(x', x_N + t)\eta(t).$$

Then $f(0) = u(x)$ and $f(t) \in W^1(\Omega)$ for $t > 0$. To compute the norm defined by (3.4), we have

$$\begin{aligned} |I_f|^2 \le \ & C \left(\int_0^\varepsilon \int_{\Omega\cap U} \mid u(x', x_N + t) \mid^2 dx \ t^{1-2\alpha} dt \right. \\ & \left. + \int_0^\varepsilon \int_{\Omega\cap U} t^{1-2\alpha} \mid \nabla u(x', x_N + t) \mid^2 dx dt \right). \end{aligned} \tag{3.5}$$

Since $1 - 2\alpha > -1$, the first integral on the right-hand side of (3.4) is bounded by $\|u\|^2_{L^2(\Omega)}$. To estimate the second integral on the right-hand side of (3.5), we first note that for $x \in \Omega \cap U$, using (3.2), there exists $C_1 > 0$,

$$\begin{aligned} \delta(x', x_N + t) &\ge C_1(x_N + t - \phi(x')) \\ &\ge C_1 t. \end{aligned}$$

Thus, after changing variables and the order of integration, we have

$$\begin{aligned} &\int_0^\varepsilon \int_{\Omega\cap U} t^{1-2\alpha} |\nabla u(x', x_N + t)|^2 dx dt \\ &\quad \le \int_{\Omega\cap U} \int_0^{C\delta(x)} t^{1-2\alpha} |\nabla u(x)|^2 dt dx \\ &\quad \le C \int_{\Omega\cap U} \delta(x)^{2-2\alpha} |\nabla u(x)|^2 dx \\ &\quad < \infty. \end{aligned}$$

This implies that $I_f < \infty$ and $u \in W^\alpha(\Omega)$. Theorem C.2 is proved.

Theorem C.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^∞ boundary and let s be a positive integer. If $u \in W_0^s(\Omega)$, then we have*

$$\delta^{-s+|\alpha|} D^\alpha u \in L^2(\Omega), \qquad \text{for every } \alpha \text{ with } |\alpha| \le s,$$

where δ is the distance function to the boundary, α is a multiindex and D^α is defined as in Appendix A.

Proof. If $f \in C_0^\infty(0,\infty)$, using Taylor's theorem, we have

$$f(x) = \frac{1}{(s-1)!}\int_0^x f^{(s)}(t)(x-t)^{s-1}dt.$$

Applying Hardy's inequality (Theorem B.8 in the Appendix), we see that

$$\begin{aligned} \left\| \frac{f(x)}{x^s} \right\|_{L^2} &\le \left\| \frac{1}{(s-1)!\; x}\int_0^x |f^{(s)}(t)|dt \right\|_{L^2} \\ &\le \frac{2}{(s-1)!}\|f^{(s)}(t)\|_{L^2}. \end{aligned}$$

Using localization and a partition of unity, we can assume that u is supported in a compact set in the upper half space $\{x = (x', x_N) \mid x_N \ge 0\}$. Applying the argument to the Taylor expansion in the x_N variable, we have for any $u \in C_0^\infty(\Omega)$,

$$\|\delta^{-s+|\alpha|} D^\alpha u\|_{L^2(\Omega)} \le C\|u\|_{W_0^s(\Omega)}.$$

The theorem follows by approximating $u \in W_0^s(\Omega)$ by functions in $C_0^\infty(\Omega)$.

Theorem C.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^∞ boundary. Let s be any positive number such that $s \ne n - 1/2$ for any $n \in \mathbb{N}$. If $u \in L^2(\Omega, loc)$ and*

$$\int_\Omega \delta^{2s}|u|^2 dV < \infty, \tag{3.6}$$

where δ is the distance function to the boundary, then $u \in W^{-s}(\Omega)$.

When $s = n - 1/2$ for some positive integer n, if we assume in addition that u is harmonic, the same statement also holds.

Proof. We first assume that s is a positive integer. For any $v \in W_0^s(\Omega)$, we have from Theorem C.3,

$$\begin{aligned} |(u,v)| &\le \|\delta^s u\|\|\delta^{-s}v\| \\ &\le C_s\|\delta^s u\|\|v\|_{W_0^s}. \end{aligned}$$

Thus, $u \in W^{-s}(\Omega)$ from definition.

For other s when $s \ne n - 1/2$, we use interpolation between $W^{-s}(\Omega)$. For $s_2 > s_1 \ge 0$, s_1, s_2 integers, if $(1-\theta)s_1 + \theta s_2 \ne n - 1/2$, then

$$[W^{-s_1}(\Omega), W^{-s_2}(\Omega)]_\theta = W^{-(1-\theta)s_1 - \theta s_2}(\Omega). \tag{3.7}$$

When $(1-\theta)s_1+\theta s_2 = n-1/2$, (3.7) no longer holds (see Lions-Magenes [LiMa 1]) and we restrict ourselves to harmonic functions.

We first prove for $s=1/2$. Using a partition of unity, we may assume that Ω is star-shaped and $0\in\Omega$. Define

$$v(x) = \int_0^1 \frac{1}{s}u(sx)ds.$$

Then v is harmonic and

$$\langle x, \nabla v(x)\rangle = \sum_{i=1}^N \int_0^1 x_i \frac{\partial u}{\partial x_i}(sx)ds = \int_0^1 \frac{\partial}{\partial s}u(sx)ds = u(x)-u(0).$$

Without loss of generality, we may assume that $u(0)=0$. We have expressed u as a linear combination of the derivatives of some harmonic function v and, from our assumption,

$$\int_\Omega \delta(x)|\langle x,\nabla v\rangle|^2 dV = \int_\Omega \delta(x)|u|^2 dV < \infty, \tag{3.8}$$

where C is some positive constant. We claim that

$$\int_\Omega \delta(x)|\nabla v|^2 dV \le C\left(\int_\Omega \delta(x)|\langle x,\nabla v\rangle|^2 dV + \int_\Omega \delta(z)|v(x)|^2 dV\right). \tag{3.9}$$

To prove (3.9), we apply the Rellich identity to the harmonic function v on the boundary $b\Omega_\eta$, where $\Omega_\eta = \{x\in\Omega \mid \delta(x)>\eta\}$ for small $\eta>0$. We have

$$\int_{b\Omega_\eta}\left(|\nabla v|^2\langle x,n\rangle - 2\langle x,\nabla v\rangle\frac{\partial u}{\partial n} - (N-2)v\frac{\partial v}{\partial n}\right)dS = 0, \tag{3.10}$$

where n is the outward normal on $b\Omega_\eta$ and dS is the surface element on $b\Omega_\eta$. Identity (3.10) follows from the equality

$$\begin{aligned}\sum_{i=1}^N \frac{\partial}{\partial x_i}&\left(|\nabla v|^2 x_i - 2\frac{\partial v}{\partial x_i}\langle x,\nabla v\rangle - (N-2)v\frac{\partial v}{\partial x_i}\right)\\ &= -2\triangle v\langle x,\nabla v\rangle - (N-2)v\triangle v = 0\end{aligned}$$

and Stokes' theorem. If η is sufficiently small, we have $\langle x,n\rangle > C_0 > 0$ for some $C_0>0$ uniformly on $b\Omega_\eta$, it follows from (3.10) that

$$\begin{aligned}C_0\int_{b\Omega_\eta}|\nabla v|^2 dS &\le \int_{b\Omega_\eta}\left|2\langle x,\nabla v\rangle\frac{\partial v}{\partial n} + (N-2)v\frac{\partial v}{\partial n}\right|dS\\ &\le \epsilon\int_{b\Omega_\eta}\left|\frac{\partial v}{\partial n}\right|^2 dS + C_\epsilon\left(\int_{b\Omega_\eta}|v|^2 dS + |\langle x,\nabla v\rangle|^2 dS\right),\end{aligned} \tag{3.11}$$

where $\epsilon > 0$. If ϵ is sufficiently small, the first term on the right-hand side of (3.11) can be absorbed by the left-hand side and we obtain

$$\int_{b\Omega_\eta} |\nabla v|^2 \leq C\left(\int_{b\Omega_\eta} |v|^2 dS + |\langle x, \nabla v\rangle|^2 dS\right). \tag{3.12}$$

Multiplying (3.12) by η and integrating over η, (3.9) is proved. Using (3.8) and (3.9), we get

$$\int_\Omega \delta(x)|\nabla v|^2 dV \leq C\int_\Omega \delta(x)|u|^2 dV < \infty. \tag{3.13}$$

It follows from Theorem C.2 that $v \in W^{\frac{1}{2}}(\Omega)$. Since for any first order derivative D with constant coefficients, we have

$$D : HW^{\frac{1}{2}}(\Omega) \to HW^{-\frac{1}{2}}(\Omega), \tag{3.14}$$

where $HW^s(\Omega) = W^s(\Omega) \cap \{u \in C^\infty(\Omega) \mid \triangle u = 0\}$. This implies that $u \in W^{-1/2}(\Omega)$. The cases for other integers can be proved similarly and this completes the proof of Theorem C.4.

We remark that (3.14) does not hold without restricting to the subspace of harmonic functions (see [LiMa 1]). The technique used in the proof of Theorem C.2 involves real interpolation, while the proof of (3.14) uses complex interpolation. We refer the reader to Jerison-Kenig [JeKe 1] and Kenig [Ken 3] for more discussion on these matters.

D. Friedrichs' Lemma

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a function with support in the unit ball such that $\chi \geq 0$ and

$$\int \chi dV = 1. \tag{4.1}$$

We define $\chi_\varepsilon(x) = \varepsilon^{-N}\chi(x/\varepsilon)$ for $\varepsilon > 0$. Extending f to be 0 outside D, we define for $\varepsilon > 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned} f_\varepsilon(x) = f * \chi_\varepsilon(x) &= \int f(x')\chi_\varepsilon(x - x')dV(x') \\ &= \int f(x - \varepsilon x')\chi(x')dV(x'). \end{aligned}$$

In the first integral defining f_ε we can differentiate under the integral sign to show that f_ε is $C^\infty(\mathbb{R}^N)$. From Young's inequality for convolution, we have

$$\| f_\varepsilon \| \leq \| f \| . \tag{4.2}$$

Since χ_ϵ is an approximation of the identity, we have $f_\varepsilon \to f$ uniformly if $f \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is a dense subset of $L^2(\mathbb{R}^N)$, this implies that

$$f_\varepsilon \to f \quad \text{in } L^2(\mathbb{R}^N) \quad \text{for every } f \in L^2(\mathbb{R}^N).$$

A very useful fact concerning approximating solutions of a first order differential operator by regularization using convolution is given by the following lemma (see Friedrichs [Fri 1] or Hörmander [Hör 2]):

Lemma D.1 (Friedrichs' Lemma). *If $v \in L^2(\mathbb{R}^N)$ with compact support and a is a C^1 function in a neighborhood of the support of v, it follows that*

$$aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \to 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0, \tag{4.3}$$

where $D_i = \partial/\partial x_i$ and $aD_i v$ is defined in the sense of distribution.

Corollary D.2. *Let*

$$L = \sum_{i=1}^{N} a_i D_i + a_0$$

be a first order differential operator with variable coefficients where $a_i \in C^1(\mathbb{R}^N)$ and $a_0 \in C(\mathbb{R}^N)$. If $v \in L^2(\mathbb{R}^N)$ with compact support and $Lv = f \in L^2(\mathbb{R}^N)$ where Lv is defined in the distribution sense, the convolution $v_\epsilon = v * \chi_\epsilon$ is in $C_0^\infty(\mathbb{R}^N)$ and

$$v_\epsilon \to v, \quad Lv_\epsilon \to f \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0. \tag{4.4}$$

Proof of Friedrichs' lemma. First note that if $v \in C_0^\infty(\mathbb{R}^N)$, we have from the discussion above,

$$D_i(v * \chi_\epsilon) = (D_i v) * \chi_\epsilon \to D_i v, \quad (aD_i v) * \chi_\epsilon \to aD_i v,$$

with uniform convergence. We claim that

$$\| aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \| \le C \| v \|, \quad v \in L^2(\mathbb{R}^N), \tag{4.5}$$

where C is some positive constant independent of ϵ and v. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, (4.3) will be proved if one can prove (4.5). To see this, we approximate v by a sequence $v_j \in C_0^\infty(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and observe that if (4.5) holds, we have

$$\begin{aligned} &\| aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \| \\ &\quad \le C(\| v - v_j \| + \| aD_i(v_j * \chi_\epsilon) - (aD_i v_j) * \chi_\epsilon \|). \end{aligned}$$

Thus, it remains to prove (4.5). Without loss of generality, we may assume that $a \in C_0^1(\mathbb{R}^N)$ since v has compact support. We have for $v \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} &aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \\ &= a(x) D_i \int v(x-y)\chi_\epsilon(y)dy - \int a(x-y)\frac{\partial v}{\partial x_i}(x-y)\chi_\epsilon(y)dy \\ &= \int (a(x) - a(x-y))\,\frac{\partial v}{\partial x_i}(x-y)\chi_\epsilon(y)dy \\ &= -\int (a(x) - a(x-y))\,\frac{\partial v}{\partial y_i}(x-y)\chi_\epsilon(y)dy \\ &= \int (a(x) - a(x-y))\,v(x-y)\frac{\partial}{\partial y_i}\chi_\epsilon(y)dy \\ &\quad - \int \left(\frac{\partial}{\partial y_i} a(x-y)\right) v(x-y)\chi_\epsilon(y)dy. \end{aligned}$$

Let M be the Lipschitz constant for a such that $|a(x) - a(x-y)| \leq M|y|$ for all $x,\ y$. We obtain

$$|aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon| \leq M \int |v(x-y)| \, (\chi_\epsilon(y) + |yD_i\chi_\epsilon(y)|) \, dy.$$

Using Young's inequality for convolution, we have

$$\begin{aligned} \| \, aD_i(v * \chi_\epsilon) - (aD_i v) * \chi_\epsilon \, \| &\leq M \, \| \, v \, \| \int (\chi_\epsilon(y) + |yD_i\chi_\epsilon(y)|) \, dy \\ &= M(1 + m_i) \, \| \, v \, \|, \end{aligned}$$

where

$$m_i = \int |yD_i\chi_\epsilon(y)| dy = \int |y(D_i\chi)(y)| dy.$$

This proves (4.5) when $v \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, we have proved (4.5) and the lemma.

Proof of the Corollary. Since $a_0 v \in L^2(\mathbb{R}^N)$, we have

$$\lim_{\epsilon \to 0} a_0(v * \chi_\epsilon) = \lim_{\epsilon \to 0} (a_0 v * \chi_\epsilon) = a_0 v \quad \text{in } L^2(\mathbb{R}^N).$$

Using Friedrichs' lemma, we have

$$Lv_\epsilon - Lv * \chi_\epsilon = Lv_\epsilon - f * \chi_\epsilon \to 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \text{as } \epsilon \to 0.$$

The corollary follows easily since $f * \chi_\epsilon \to f$ in $L^2(\mathbb{R}^N)$.

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Table of Notation

The following is a partial list of notation. The section number refers to the section where the notation is first encountered or defined.

General

$\mathbb{C}^n$	n-dimensional complex Euclidean space — 1.1
D, Ω	domain in $\mathbb{C}^n$ or a complex manifold (open connected subset) — 1.1, 5.1
bD	boundary of D — 1.1
$A \setminus B$	elements in set A but not in set B — 1.2
$\mathbb{RP}^n$	real projective space — 1.2
$\mathbb{CP}^n$	complex projective space — 1.2
J	complex structure — 1.3
$\mathbb{C}T_p(M)$	$= T_p(M) \otimes_{\mathbb{R}} \mathbb{C}$: complexified tangent space at p on a complex manifold — 1.3; on a CR manifold — 7.1
$T_p^{1,0}(M)$, $T_p^{0,1}(M)$	tangent vectors of type (1,0) and (0,1) at p on a complex manifold — 1.3; on a CR manifold — 7.1
$\mathbb{C}T_p^*(\mathbb{C}^n)$	dual space of $\mathbb{C}T_p(\mathbb{C}^n)$ — 1.3
$\Lambda_p^{1,0}(\mathbb{C}^n)$, $\Lambda_p^{0,1}(\mathbb{C}^n)$	Space of (1,0)- and (0,1)-forms at p — 1.3
dz_j, $d\bar{z}_j$	(1,0)- and (0,1)-form — 1.3
$T_p(M)$	tangent space at p — 1.4
$T_p^*(M)$	cotangent space at p — 1.4
$\wedge$	wedge product — 1.4, 1.5
$\Lambda^r(M)$	vector bundle of degree r on M — 1.4
$\Lambda^{p,q}(M)$	vector bundle of bidegree (p,q) on a complex manifold — 1.4; on a CR manifold — 7.2
$A \lesssim B$	$A \leq CB$ for some constant $C > 0$ — 2.1
$P(\zeta; r)$	a polydisc with center $\zeta = (\zeta_1, \cdots, \zeta_n)$ and multiradii $r = (r_1, \cdots, r_n)$ in $\mathbb{C}^n$ — 2.1
$\sum'$	sum over increasing multiindices — 2.1
$\mathcal{L}_p(r, \cdot)$	Levi form of the function r at p — 3.3
$T_p^{1,0}(bD)$, $T_p^{0,1}(bD)$	tangent vector to bD at p of type (1,0) and (0,1) — 3.3
$d_D(z)$	Euclidean distance from $z \in D$ to bD — 3.4
$dist(z, bD)$	Euclidean distance from $z \in \mathbb{C}^n \setminus bD$ to bD — 3.4
$D \subset\subset D'$	D is relatively compact in D', i.e., $\overline{D} \subset D'$ — 3.4
dV	volume element in $\mathbb{C}^n$ — 3.4
$\widehat{K}_D$	holomorphically convex hull of K in D — 3.5
dz^I	$= dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ for $I = (i_1, \cdots, i_p)$ — 4.2
$d\bar{z}^J$	$= d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ for $J = (j_1, \cdots, j_q)$ — 4.2
$\vee$	interior product — 4.2

dS	surface element on the boundary bD — 4.2
ϵ^A_B	sign of permutation — 4.3
ω^I	$= \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$ for $I = (i_1, \cdots, i_p)$ — 4.3
$\overline{\omega}^J$	$= \overline{\omega}^{j_1} \wedge \cdots \wedge \overline{\omega}^{j_q}$ for $J = (j_1, \cdots, j_q)$ — 4.3
f^τ	complex tangential part of f — 4.3
f^ν	complex normal part of f — 4.3
$\tilde{f}$	partial (or tangential) Fourier transform of f — 5.2
$O(\|A\|)$	terms bounded by $C\|A\|$ for some constant $C > 0$ — 5.2
supp f	support of f — 5.2
f'	complex Jacobian of f — 6.3
$T^{*1,0}(M)$, $T^{*0,1}(M)$	dual bundles of $T^{1,0}(M)$ and $T^{0,1}(M)$ — 7.2
Ω_n	Siegel upper half space — 7.3
δ_{jk}	Kronecker delta — 7.3
$\mathbb{H}_n$	Heisenberg group of order $n-1$ — 7.3, 10.1
$\widehat{f}$	Fourier transform of f — 7.4
$Aut(\Omega_n)$	automorphism group on Ω_n — 10.1
$\mathbb{H}_{n,k}$	generalized Heisenberg group — 10.1
$[\widehat{d\overline{z}_j}]$	$= d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_j} \wedge \cdots \wedge d\overline{z}_n$ — 11.1
$< A, B >$	pairing between vectors A, B in $\mathbb{C}^n$ — 11.1
$< A, d\zeta >$	pairing between vector A and 1-form $d\zeta$ — 11.1

Spaces of Functions (or Forms) and Norms

$\mathcal{O}(D)$	holomorphic functions on D — 1.1
$C^k(D)$	$k \in \mathbb{N}$ or $k = \infty$: k times continuously differentiable functions on D — 1.1
$C^k(\overline{D})$	$k \in \mathbb{N}$ or $k = \infty$: k times continuously differentiable functions on $\overline{D}$, i.e., restriction of $C^k(\mathbb{C}^n)$ to $\overline{D}$ — 2.1
$C^\infty_0(D)$	C^∞ functions with compact suuprot in D — 2.1
$C(\overline{D})$	continuous functions on $\overline{D}$ — 2.1
$C^\lambda(D)$, $C^{k+\lambda}(D)$	$0 < \lambda < 1$, $k \in \mathbb{N}$: Hölder continuous functions of order λ and $k + \lambda$ on D — 2.1
$\Lambda^1(D)$	Lipschitz continuous functions on D — 2.1
$L^2(D)$	square integrable functions on D — 4.2
$\\| \ \\|$	L^2 norm in $L^2(D)$ — 4.2
$C^k_{(p,q)}(D)$	(p,q)-forms on D with coefficients in $C^k(D)$ — 4.2
$C^k_{(p,q)}(\overline{D})$	(p,q)-forms on D with coefficients in $C^k(\overline{D})$ — 4.2
$L^2_{(p,q)}(D)$	(p,q)-forms on D with coefficients in $L^2(D)$ — 4.2
$L^2(D, \phi)$	weighted L^2 space with weighted $e^{-\phi}$ — 4.2
$\\| \ \\|_\phi$	weighted L^2 norm in $L^2(D, \phi)$ — 4.2
$(\ , \ )_\phi$	weighted L^2 inner product in $L^2(D, \phi)$ — 4.2
$L^2(D, loc)$	locally square interable functions; functions which are square integrable on any compact subset of D — 4.2
$L^2_{(p,q)}(D, \phi)$	(p,q)-forms on D with coefficients in $L^2(D, \phi)$ — 4.2

$L^2_{(p,q)}(D, loc)$	(p, q)-forms on D with coefficients in $L^2(D, loc)$ — 4.2
$\| \ \|_s$	Sobolev norm of order s — 4.5
$W^s(D)$	Sobolev s-space — 4.5
$W^s(D, loc)$	functions in $W^s(D')$ for any relatively compact subset D' of D — 4.5
$W^s_{(p,q)}(D)$	(p, q)-forms on D with coefficients in $W^s(D)$ — 5.1
$W^s_{(p,q)}(D, loc)$	(p, q)-forms on D with coefficients in $W^s(D, loc)$ — 5.1
$W^s_0(D)$	closure of $C^\infty_0(D)$ in $W^s(D)$ — 5.1
$\|\|\| \ \|\|\|_s$	tangential Sobolev norm of order s — 5.2
$\mathcal{H}_{(p,q)}(\Omega)$	harmonic (p, q)-forms on a hermitian manifold Ω — 5.3
$\mathcal{H}(D)$	square integrable holomorphic functions on D — 6.1
$\mathcal{H}^s(D)$	$= W^s(D) \cap \mathcal{O}(D)$ — 6.3
$\mathcal{E}^{p,q}(M)$	smooth (p, q)-forms on a CR manifold — 7.2
$\mathcal{S}$	Schwartz space in $\mathbb{R}^n$ — 8.1
$\mathcal{F}_{(p,q)}(M)$	(p, q)-forms on a CR manifold M with coefficients in $\mathcal{F}$ where $\mathcal{F}$ is any of the function spaces $L^2(M)$, $W^s(M)$ and $C^k(M)$ — 8.3
$\mathcal{H}^b_{(p,q)}(M)$	harmonic (p, q)-forms on a CR manifold M — 8.4
$\|\|\|\Theta^m f\|\|\|_s$	weighted tangential Sobolev norm of order s — 9.3
$H^2(\Omega_n)$	Hardy space on Ω_n — 10.2
$\mathcal{O}(K)$	holomorphic functions on neighborhoods of the compact set K — 11.1
$L^p(\omega_\delta)$	$1 < p < \infty$: p-th power integrable functions on ω_δ — 11.5
$\| \ \|_{L^p(\omega_\delta)}$	L^p norm in $L^p(\omega_\delta)$ — 11.5
$\mathcal{F}_{(p,q)}(\omega_\delta)$	(p, q)-forms on ω_δ with coefficients in $\mathcal{F}$ where $\mathcal{F}$ is any of the function spaces $L^p(\omega_\delta)$, $L^p(\omega_\delta, loc)$, $W^s(\omega_\delta)$, $W^s(\omega_\delta, loc)$, $C^k(\omega_\delta)$ and $C^k(\overline{\omega}_\delta)$ — 11.5, 11.6
$\mathcal{H}_b(\omega_\delta)$	$= L^2(\omega_\delta) \cap Ker(\overline{\partial}_b)$ — 11.6
$\mathcal{H}(M)$	$= L^2(M) \cap Ker(\overline{\partial}_b)$ on a CR manifold M — 12.2
$\mathcal{P}_k$	homogeneous polynomials of degree k in $\mathbb{R}^n$ — 12.3
$\mathcal{HP}_k$	solid spherical harmonics (homogeneous harmonic polynomials of degree k) — 12.3
$\mathcal{SP}_k$	$= \mathcal{HP}_k\|_{S^{n-1}}$: surface spherical harmonics — 12.3
$\mathcal{HP}^{p,q}_k$	elements in $\mathcal{HP}_k$ generated by $z^\alpha \overline{z}^\beta$ with $\|\alpha\| = p$, $\|\beta\| = q$ and $p + q = k$ — 12.4
$\mathcal{SP}^{p,q}_k$	$= \mathcal{HP}^{p,q}_k\|_{S^3}$ — 12.4

Operators and Related Definitions

d	exterior derivative — 1.5
$\triangle$	Laplacian — 4.2
$\overline{\partial}$	Cauchy-Riemann operator in $\mathbb{C}^n$ or on a complex manifold — 1.1, 1.5, 4.2
$Dom(T)$	domain of the operator T — 4.1

$Ker(T)$, $\mathcal{R}(T)$	kernel and range of the operator T — 4.1
$\overline{\mathcal{R}(T)}$	closure of $\mathcal{R}(T)$ — 4.1
ϑ	formal adjoint of $\overline{\partial}$ — 4.2
ϑ_ϕ	formal adjoint of $\overline{\partial}$ under the $L^2(D,\phi)$ norm — 4.2
$\overline{\partial}^*$	L^2 adjoint of $\overline{\partial}$ — 4.2
$\overline{\partial}^*_\phi$	L^2 adjoint of $\overline{\partial}$ with the weighted norm $\|\ \|_\phi$ — 4.2
$\sigma(\vartheta, d\rho)$	symbol of ϑ in the direction of $d\rho$ — 4.2
$\Box$	$=\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$: $\overline{\partial}$-Laplacian ($\overline{\partial}$-Neumann problem) — 4.2
$\mathcal{D}^l_{(p,q)}$	$=C^l_{(p,q)}(\overline{D})\cap Dom(\overline{\partial}^*)$ — 4.3
$\mathcal{D}_{(p,q)}$	$=C^\infty_{(p,q)}(\overline{D})\cap Dom(\overline{\partial}^*)$ — 4.3
$\mathcal{D}^\phi_{(p,q)}$	$=C^\infty_{(p,q)}(\overline{D})\cap Dom(\overline{\partial}^*_\phi)$ — 4.3
$\delta^\phi_j u$	$=e^\phi \frac{\partial}{\partial z_j}(e^{-\phi}u)$ — 4.3; $=e^\phi L_j(e^{-\phi}u)$ — 5.3
$N_{(p,q)}$	$=N$: $\overline{\partial}$-Neumann operator on D in $\mathbb{C}^n$ — 4.4; on a hermitian manifold Ω — 5.3
$\mathcal{H}_{(p,0)}(D)$	$=L^2_{(p,0)}(D)\cap Ker(\overline{\partial})$ — 4.4
$H_{(p,0)}$	orthogonal projection from $L^2_{(p,0)}(D)$ onto $\mathcal{H}_{(p,0)}(D)$ — 4.4
P	Bergman projection — 4.4
N_ϕ, N_t	weighted $\overline{\partial}$-Neumann operator — 4.5, 6.1
Λ^s_t	a special tangential pseudodifferential operator of order s — 5.2
τ	projection from $\Lambda^{p,q}(\mathbb{C}^n)$ to $\Lambda^{p,q}(M)$ — 7.2
$\overline{\partial}_b$	tangential Cauchy-Riemann operator on a CR manifold — 7.2
$\pi_{p,q}$	projection from $\Lambda^{p+q}\mathbb{C}T^*(M)$ onto $\Lambda^{p,q}(M)$ — 7.2
Λ^s	a special pseudodifferential operator of order s — 8.1
$\overline{\partial}^*_b$	L^2 adjoint of $\overline{\partial}_b$ — 8.3
$\Box_b$	$=\overline{\partial}_b\overline{\partial}^*_b + \overline{\partial}^*_b\overline{\partial}_b$: $\overline{\partial}_b$-Laplacian — 8.3
N_b	inverse operator for $\Box_b$ on a compact CR manifold (or sometimes called the $\overline{\partial}_b$-Green operator) — 8.4
$H^b_{(p,q)}$	orthogonal projection from $L^2_{(p,q)}(M)$ onto $\mathcal{H}^b_{(p,q)}(M)$ — 8.4
S, S_b	Szegö projection — 8.4; on ω_δ — 11.6
$\star$	Hodge star operator — 9.1
$\widetilde{S}$	$=H^b_{(n,n-1)}$ — 9.4
$\mathcal{L}_0$	sub-Laplacian on a stratified Lie group — 10.1
$\mathcal{L}_\alpha$	$=\mathcal{L}_0 + i\alpha T$ — 10.1
$\overline{\partial}_{\zeta,z}$	$=\overline{\partial}_\zeta + \overline{\partial}_z$ — 11.1
$\sigma(\vartheta_b, dr)$	symbol of ϑ_b in the direction of dr — 11.6
$\mathcal{N}_b$	$\overline{\partial}_b$-Neumann operator — 11.6

Kernel Functions and Integral Operators

$1/(2\pi iz)$	Cauchy kernel (fundamental solution for $\overline{\partial}$ in $\mathbb{C}$) — 2.1, 11.1
$B(\zeta, z)$	Bochner-Martinelli kernel — 2.2

$K_D(z,w)$	Bergman kernel function on D — 6.3
$K_\lambda(z,w)$	weighted Bergman kernel function — 6.5
Φ_α	fundamental solution for $\Box_b$ on the Heisenberg group — 10.1
$S(z,w)$	Cauchy-Szegö kernel — 10.2
Φ	relative fundamental solution for $\Box_b^0$ on the Heisenberg group — 10.3
$e(z)$	radially symmetric fundamental solution for $\triangle$ in $\mathbb{C}^n$ — 11.1
$\Omega(G)$	(Cauchy-Fantappie) kernel constructed from the map G — 11.1
$\Omega(G^1,\cdots,G^m)$	$=\Omega^{1\cdots m}$ or $\Omega_q^{1\cdots m}$: kernel constructed from maps $G^1,\cdots,G^m$ — 11.1
$B_q(\zeta,z)$	Bochner-Martinelli-Koppelman kernel on forms — 11.1
T_q	integral solution operator for $\overline{\partial}$ on a convex domain D — 11.2
H_q, R_q	integral solution operator for $\overline{\partial}_b$ on a strictly convex boundary — 11.3
S_q, $\widetilde{S}_q$	integral solution operator for $\overline{\partial}_b$ on a CR manifold with boundary — 11.4, 11.5

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