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Foliations in Cauchy-Riemann Geometry

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Contents

Preface

The present monograph is an attempt to a better understanding of an interdisciplinary question, namely the impact of foliation theory on the geometry and analysis on CR manifolds. To start with, any Levi-flat CR manifold M carries a complex foliation $\mathcal{F}$ (the *Levi foliation*) tangent to the null space of the Levi form of the manifold. At least in the real analytic case, if M is embedded then $\mathcal{F}$ extends to a holomorphic foliation of an open neighborhood of M (*Rea's theorem*, [**203**]). Complex foliations occur in a natural way on certain nondegenerate CR manifolds, as well. To give a simple example, R. Penrose's manifold $\mathbb{P}(\mathbb{T}_0)$ (the boundary of the manifold $\mathbb{P}(\mathbb{T}_+)$ of all right-handed spinning photons, cf. [**199**]) is a nondegenerate CR manifold of hypersurface type foliated by $\mathbb{C}P^1$'s and this situation generalizes to C. Le Brun's twistor CR manifolds (cf. [**165**]-[**166**]). As shown in [**84**], there are also natural CR analogues of complex Monge-Ampére foliations (in the sense of [**30**], for instance) occurring on strictly pseudoconvex CR manifolds. Each leaf of such a tangential Monge-Ampére foliation is a CR manifold and the inclusion in the ambient space is a pseudohermitian immersion. Finally, let us mention that each nondegenerate CR manifold carries a flow defined by its contact vector field. This is evidence enough to the interrelation between foliation theory and CR geometry, and that an overall use of the former is liable to clear up certain questions in complex analysis. For instance, let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with real analytic boundary $\partial\Omega$. Let $\mathcal{O}(\overline{\Omega})$ be the algebra of functions on $\overline{\Omega}$ which admit a holomorphic extension to some neighborhood of $\overline{\Omega}$. Let $M \subset \partial\Omega$ be a real analytic submanifold which is not $\mathbb{C}$-tangent at any of its points. By a result of L. Boutet de Monvel and A. Iordan (cf. [**56**]) M is locally a maximum modulus set for $\mathcal{O}(\overline{\Omega})$ (in the sense of T. Duchamp and E.L. Stout, [**93**]) if and only if $\mathcal{L}(X, Y)$ is real valued for any sections X, Y in $L = T(M) \cap H(\partial\Omega)$. Here $\mathcal{L}$ is the Levi form of $\partial\Omega$ and $H(\partial\Omega)$ is its maximal complex distribution. If this is the case then L is completely integrable and gives rise to a $\mathbb{C}$-tangent foliation $\mathcal{F}$ of M of codimension one and the paper [**22**] studies the interplay between the properties of $\mathcal{F}$ and the geometry of the second fundamental form of M in $\partial\Omega$. M turns out to be a Levi flat contact CR submanifold of $\partial\Omega$ and $\mathcal{F}$ is its Levi foliation. When M is minimal $\mathcal{F}$ is harmonic.

Let us add that, besides from the very interest in interdisciplinary problems, in a series of papers (cf. [**16**], [**19**] and [**21**]) the first two authors developed an idea of E.M. Chirka, [**67**], regarding foliations with transverse CR structure (which contain the class of CR manifolds as the special case of transversally CR foliations by points) which led to Chapter 6 of this monograph.

Sections 1.1 and 1.2 review the notions of foliation theory needed through the text. We only sketch the essentials, as many monographs on the subject have been available for quite a few years (such as [**179**], or [**243**], which are the most frequently referred to).

The next seven chapters form the main core of this book. The case of foliated CR manifolds is considered in Chapter 2. Sections 2.3 to 2.5 are imitative of P. Tondeur's exposition of the geometry of foliations on Riemannian manifolds, cf. [**243**], p. 47-73, and the similarity comes from the fact that in the nondegenerate case CR manifolds possess a canonical metric (the *Webster metric*) and connection (the *Tanaka-Webster connection*) of which the latter resembles the Chern connection in Hermitian geometry and the Levi-Civita connection in Riemannian geometry.

Chapter 3 is dedicated to Levi foliations and their holomorphic extendibility. We give a proof of a beautiful result referred to as *Rea's theorem*. It is based on a theorem by F. Severi and G. Tomassini (cf. [**219**] and [**242**]) about holomorphic extension of CR functions in the real analytic case. There are many other CR extension theorems available in today's mathematical literature (cf. [**50**] and references therein) yet it seems to the authors that Rea's is the only attempt (cf. [**203**]) to apply a CR extension result in order to get a holomorphic extension of a Levi (or semi-holomorphic) foliation.

Related to Rea's theorem we present the solution (due to D.E. Barrett, [**28**]) to the problem of the existence of a pluriharmonic defining function for a Levi-flat real analytic hypersurface in a complex manifold. Next, we exhibit a characterization of Levi flatness of real analytic hypersurfaces in $\mathbb{C}^n$ in terms of *holomorphic degeneracy*, cf. Theorem 3.22 in Section 3.4 (due to N.K. Stanton, [**227**]).

An active research field in complex analysis (in several complex variables) is that related to the problem of global regularity of the Neumann operators N_q, $1 \leq q \leq n$, and of the Bergman projections P_q, $0 \leq q \leq n$, for a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Precisely, the question is whether N_q and P_q are continuous on the space $W^s_{(0,q)}(\Omega)$, $s \geq 0$, of all $(0,q)$-forms with coefficients in the Sobolev space $W^s(\Omega)$, $s \geq 0$ (cf. e.g. [**48**]). The state of the art is represented by Theorem 3.32 (due to H.P. Boas and E.J. Straube, [**49**]) in Section 3.5 of this monograph. The estimates leading to the result in Theorem 3.32 were known (by a result of D. Catlin, [**61**]) at the points of finite type, yet required a new technique, based on the existence of complex vector fields commuting approximately with $\overline{\partial}$ (cf. Definition 3.33) on the set $K \subset \partial\Omega$ of all boundary points of infinite type. Such vector fields were shown to exist when Ω admits a plurisubharmonic defining function (cf. Definition 3.30), a fact which led to Theorem 3.31 (due again to H.P. Boas et al., [**48**]). When the set K of all infinite type points is contained in a real submanifold $M \subset \partial\Omega$ of the boundary which is tangent (i.e. $T(M) \subset \text{Null}(G_\theta)$) to the Levi null distribution (e.g. when $K = \overline{\mathring{K}}$ and the Levi form of $\partial\Omega$ vanishes at each point of $\mathring{K}$) the beautiful (from a differential geometric viewpoint) finding by H.P. Boas et al., [**49**] (and further examined by E.J. Straube and M.K. Sucheston, [**233**]) is the existence of a de Rham cohomology class $a(M) \in H^1(M, \mathbb{R})$ (the *D'Angelo class*, under the terminology adopted in this monograph) which is an obstruction to the existence of the special vector fields mentioned above (cf. Theorem 3.36). Section 3.5 concludes with a discussion of the D'Angelo class within foliation theory (i.e. the relationship among $a(M)$ and the infinitesimal holonomy of the leaf M of the Levi foliation on $\mathring{K}$) and a few open problems.

Chapter 4 reports on the known results about the nonexistence of Levi flat CR submanifolds in a complex projective space, such as Y-T. Siu's result (cf. [**221**]-[**222**]) with the lower differentiability requirements due to J. Cao and M-C. Shaw and L. Wang, [**60**] (cf. Theorem 4.1 in Section 4.1 of this book), the result of L. Ni and J. Wolfson, [**188**] (based on a Lefschetz type result for CR submanifolds of a Kählerian manifold of positive holomorphic bisectional curvature, established by themselves, and the classical theorem of A. Haefliger, [**131**], on the inexistence of real analytic codimension one foliations on compact simply connected manifolds), and the purely differential geometric approach of M. Djorić and M. Okumura, [**81**].

Chapter 5 is about foliations with tangential CR structure i.e. each of whose leaves is a CR manifold. We look at foliations by level hypersurfaces of the defining function of a strictly pseudoconvex domain in $\mathbb{C}^n$ such as occurring in C.R. Graham and J.M. Lee's paper [**124**] (and studied by them in connection with the Dirichlet problem for certain degenerate Laplacians of which the prototype is the Bergman Laplacian on the unit ball in $\mathbb{C}^n$). We give a new axiomatic description of the canonical connection there (the *Graham-Lee connection*) and use it to look at the boundary values of a Yang-Mills field in a Hermitian holomorphic vector bundle $\pi : F \to \Omega$ over a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ (cf. [**26**]). Precisely we endow Ω with the Bergman metric and consider the Dirichlet problem for the Yang-Mills equations

$$\delta^D R^D = 0 \quad \text{in } \Omega, \quad D = D^0 \quad \text{on } \partial\Omega, \tag{0.1}$$

where the boundary data D^0 is a C^∞ Hermitian connection in the Hermitian CR-holomorphic vector bundle $E = \pi^{-1}(\partial\Omega) \to \partial\Omega$. It is then shown that the boundary values D^0 of a solution D to (0.1) must be a pseudo Yang-Mills field on $\partial\Omega$ (cf. our Theorem 5.22). Section 5.6 is based on our own work on tangential Monge-Ampére foliations (cf. [**84**] and fitting into the theory of pseudohermitian immersions, cf. also [**89**]).

Chapters 6 to 8 are based on work on transversally CR foliations by the first two authors (cf. *op. cit.*). Chapter 6 is devoted to the basics while Chapters 7 and 8 present two main applications. The first regards the interrelation between G. Gigante and G. Tomassini's theory of CR Lie algebras (cf. [**116**]) and F. Fedida's $\mathcal{G}$-Lie foliations (cf. [**103**]) and includes a homotopy classification of transverse f-structures. The second is devoted to solving a transverse Beltrami equation, which is a foliated analogue of the Beltrami equation in the work of A. Korányi and H.M. Reimann, [**159**]. The effect is producing foliated quasiconformal mappings (cf. E. Barletta, [**16**]). These results extend A. Korányi and H.M. Reimann's considerations - originally holding only on strictly pseudoconvex CR manifolds - to certain degenerate CR manifolds where the degeneracy may be 'factored out' by an algebraic process leading to a strictly pseudoconvex transversally CR foliation. The authors hope that Section 6.4 may contribute to a better understanding of the features of degenerate CR manifolds.

At least for compact Hausdorff foliations i.e. foliations with all leaves compact and the leaf space Hausdorff, the leaf space has (by a result of J. Girbau and M. Nicolau, [**120**], relying itself on a result by D.B.A. Epstein, [**100**]) a natural structure of an *orbifold* (or *V-manifold* in the terminology of I. Satake, [**213**], to whom the notion is due). If this is the case a given transverse CR structure induces a CR structure (in the sense of Chapter 11) on the leaf space, the latter becoming a *CR orbifold*. Chapter 11 aims to a motivation of the need for a theory of CR orbifolds and states some open problems. On the other hand, there is a growing theory of orbifolds, among whose contributors one finds W.L. Baily, [**8**]-[**10**], J.E. Borzellino, [**51**], J.E. Borzellino and B.G. Lorica, [**53**], J.E. Borzellino and S-H. Zhu, [**52**], J.E. Borzellino and V. Brunsden, [**54**], M. Carlotti, [**62**]-[**63**], J. Girbau and M. Nicolau, [**120**], T.D. Jeffres, [**141**], H. Kitahara, [**154**], L-K. Koh, [**172**], T. Shioya, [**220**], and I. Satake himself, [**213**]-[**215**], but to the knowledge of the authors no monograph is available on this subject except for a portion of [**239**], confined to the 3-dimensional case, and of J.E. Borzellino's Ph.D. thesis, [**51**]. There are many differences in style and notations between the above quoted papers and also some inadequacies (for instance [**62**] *postulates* the existence of the monomorphism η while that may be proved, cf. Section 9.3 of this monograph). We choose to expose carefully the basics of the theory of orbifolds in Chapter 9, following mainly the paper [**120**] and hoping to remedy to the mentioned inadequacies and hinting to a further development of differential geometry and analysis on CR orbifolds. Ending these comments, we would like to mention the work by Y-J. Chiang, [**67**], on harmonic maps from a Riemannian orbifold to an ordinary Riemannian manifold (and showing that in the homotopy class of a map of a Riemannian orbifold into a Riemannian manifold of negative sectional curvature there is a harmonic representative). Y-J. Chiang's result is generalized by the work of A. El Kacimi-Alaoui and E.G. Gomez's Theorem 6 in [**148**], p. 121, as $W/\mathrm{SO}(q)$ (where W is the base of the fibration giving rise to the basic foliation associated with the lifted foliation, cf. our section 1.2) is not an orbifold unless the action of $\mathrm{SO}(q)$ on W is locally free. This means that the open problem (of which only the local part is dealt with in Section 11.5) regarding the existence of a parametrix for the Kohn-Rossi operator on a CR orbifold may find its proper and more general setting in a theory of *transversally subelliptic operators* eventually parallelling A. El Kacimi-Alaoui's work [**145**].

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APPENDIX A

Holomorphic bisectional curvature

The scope of this appendix is to collect the known results on Kähler manifolds of nonnegative holomorphic bisectional curvature, cf. [**122**].

Let V be a Kählerian manifold, of complex dimension ν. Let J denote the complex structure on V. Let g be a Kähler metric on V and R the curvature tensor field of (V, J, g). A *2-plane* at $x \in V$ is a 2-dimensional subspace $\sigma \subset T_x(V)$. A 2-plane σ at x is *J-invariant* if $J_x(\sigma) = \sigma$ and the set of all J-invariant 2-planes is the total space of a holomorphic bundle $\mathbb{C}P^{\nu-1} \to G_2(M) \to M$ (the *Grassmann bundle*). Given two J-invariant 2-planes $\sigma \subset T_x(V)$ and $\sigma' \subset T_x(V)$ the *bisectional curvature* $H(\sigma, \sigma')$ is defined by

$$H(\sigma, \sigma') = g_x(R_x(Y, J_xY)J_xX\ ,\ X)$$

where $X \in \sigma$ and $Y \in \sigma'$ are unit tangent vectors. The definition of $H(\sigma, \sigma')$ doesn't depend upon the choice of unit vectors in σ and σ'. Holomorphic bisectional curvature generalizes holomorphic sectional curvature (for $H(\sigma, \sigma)$ is nothing but the holomorphic sectional curvature of $\sigma \in G_2(M)$, as introduced for instance in [**155**], Vol. II, p. 165) and for a complex space form $V^\nu(c)$, i.e. a Kählerian manifold endowed with a Kähler metric of constant holomorphic sectional curvature c, the holomorphic bisectional curvature isn't constant but rather $H(\sigma, \sigma')$ lies between $c/2$ and c (the value $c/2$ is reached when σ and σ' are orthogonal while the value c is reached when $\sigma = \sigma'$). A generalization of a result by T. Frankel, [**110**] (requiring positive holomorphic sectional curvature) may be stated as

THEOREM A.1. (S.I. Goldberg & S. Kobayashi, [**122**])
Let V be a compact connected complex ν-dimensional Kählerian manifold endowed with a Kähler metric of positive holomorphic bisectional curvature and let M and N be two compact complex submanifolds of V. If $\dim_{\mathbb{C}} M + \dim_{\mathbb{C}} N \geq \nu$ then $M \cap N \neq \emptyset$.

Similarly, the proof of a result by A. Andreotti & T. Frankel (cf. Theorem 3 in [**110**]) may be easily adapted to show that *any compact Kähler surface with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}P^2$*. It is also known that

THEOREM A.2. (S.I. Goldberg & S. Kobayashi, [**122**])
i) *A complex ν-dimensional compact connected Kähler manifold with an Einstein metric of positive holomorphic bisectional curvature is globally isometric to $\mathbb{C}P^\nu$ with the Fubini-Study metric.*
ii) *A complex ν-dimensional compact connected Kähler manifold of positive holomorphic bisectional curvature and constant scalar curvature is holomorphically isometric to $\mathbb{C}P^\nu$.*

Theorem A.2 extends previous results of M. Berger, [**38**], and R.L. Bishop & S.I. Goldberg, [**42**]. As well as in the case of a Kähler manifold of positive holomorphic sectional curvature (cf. [**41**])

THEOREM A.3. (S.I. Goldberg & S. Kobayashi, [**122**])
The second Betti number of a compact connected Kähler manifold V of positive holomorphic bisectional curvature is $b_2(V) = 1$.

Let V be an irreducible[1] compact complex ν-dimensional Kählerian manifold of non-negative holomorphic bisectional curvature, that is $H(\sigma, \sigma') \geq 0$. For any $Z \in T(V)$, $Z \neq 0$, we set

$$H_Z(X,Y) := g(R(Z, JZ)X, JY), \quad X, Y \in T(V).$$

Then H_Z is positive semi-definite. Let $\mathcal{N}_Z$ be the corresponding null distribution i.e.

$$\mathcal{N}_Z = \{X \in T(V) : H_Z(X,Y) = 0, \quad \forall\, Y \in T(V)\}.$$

Clearly $\mathcal{N}_Z$ is J-invariant and $(\mathcal{N}_Z)_x$ is determined by Z_x for any $x \in V$. For each $v \in T_x(V) \setminus \{0\}$ let us set $\operatorname{cdim} \mathcal{N}_v := \nu - \dim_{\mathbb{C}} (\mathcal{N}_Z)_x$ where $Z \in T(V)$ is a vector field such that $Z_x = v$.

DEFINITION A.4. The *complex positivity* of V is defined by

$$\ell(x) = \inf\{\operatorname{cdim} \mathcal{N}_v : v \in T_x(V) \setminus \{0\}\}.$$

for any $x \in V$. □

By a result of N. Mok, [**177**], the complex positivity $\ell(x)$ doesn't depend on the point x. The complex positivity ℓ was computed by M. Kim & J. Wolfson, [**153**], for all compact Hermitian symmetric spaces. In particular

V	ℓ	$\mathrm{Sp}(r)/\mathrm{U}(r)$	r
$\mathbb{C}P^\nu$	ν	$\mathrm{SO}(2r)/\mathrm{U}(r)$	$2r-3$
$\mathbf{Gr}_p(\mathbb{C}^{p+q})$	$p+q-1$	$E_6/(\mathrm{Spin}(10) \times T^1)$	11
$\mathbf{Gr}_2(\mathbb{R}^{p+2})$	$p-1$	$E_7/(E_6 \times T^1)$	17

The recalled notions are needed in the discussion of the Lefschetz type results in Section 4.2 of this monograph.

[1] As a Riemannian manifold, cf. e.g. [**155**], Vol. I, p. 179.

APPENDIX B

Partition of unity on orbifolds

Let $(B, \mathcal{A})$ be a C^∞ orbifold.

DEFINITION B.1. A continuous map $f : B \to N$ of B into an ordinary C^∞ manifold N is a C^∞ *map* if, for any l.u.s. $\{\Omega, G, \varphi\} \in \mathcal{A}$, the map $f_\Omega : \Omega \to N$ given by $f_\Omega = f \circ \varphi$ is C^∞ differentiable. □

Let $f : B \to N$ be a C^∞ map. Then $f_{\Omega'} \circ \lambda = f_\Omega$ for any injection λ of $\{\Omega, G, \varphi\}$ into $\{\Omega', G', \varphi'\}$ (as $f_{\Omega'} \circ \lambda = (f \circ \varphi') \circ \lambda = f \circ \varphi = f_\Omega$). In particular, each f_Ω is G-invariant. Here we adopted the following

DEFINITION B.2. Given a l.u.s. $\{\Omega, G, \varphi\} \in \mathcal{A}$, a map $h : \Omega \to N$ is *G-invariant* if $h \circ \sigma = h$ for any $\sigma \in G$. □

With each map $F : \Omega \to N$ one may associate the G-invariant map $F^G : \Omega \to N$ given by

$$F^G = \frac{1}{n(G)} \sum_{\sigma \in G} F \circ \sigma$$

where $n(G)$ is the order of the group G.

DEFINITION B.3. F^G is called the *G-average* of F. □

If F is G-invariant then $F^G = F$. Any G-invariant C^∞ map $F : \Omega \to N$ defines a C^∞ map $f : U \to N$ on the support U of $\{\Omega, G, \varphi\}$, where U is thought of as a C^∞ orbifold with the orbifold structure induced from B (set $f(p) = F(x)$ for just any $x \in \varphi^{-1}(p)$). If $F : \Omega \to N$ is an arbitrary C^∞ map (not necessarily G-invariant) then the G-average of F does the same job (i.e. F^G induces a C^∞ map $f : U \to N$ as before).

DEFINITION B.4. Let $F : \Omega \to N$ be a G-invariant map. Let $\{\Omega', G', \varphi'\}$ be another l.u.s. of B of support $U' \subseteq U$. The *restriction* F' of F to $\{\Omega', G', \varphi'\}$ is built as follows. Let $\lambda : \{\Omega', G', \varphi'\} \to \{\Omega, G, \varphi\}$ be an injection and set $F' = F \circ \lambda$. □

The definition of F' does not depend upon the choice of injection. Indeed, if μ is another injection of $\{\Omega', G', \varphi'\}$ into $\{\Omega, G, \varphi\}$ then there is a unique $\sigma_1 \in G$ such that $\mu = \sigma_1 \circ \lambda$ and one has

$$F \circ \mu = F \circ (\sigma_1 \circ \lambda) = F \circ \lambda$$

by the G-invariance of F. The restriction of F to $\{\Omega', G', \varphi'\}$ is G'-invariant. Indeed

$$F' \circ \sigma' = (F \circ \lambda) \circ \sigma' = F \circ (\eta(\sigma') \circ \lambda) = F \circ \lambda = F'$$

for any $\sigma' \in G'$ (again by the G-invariance of F) where $\eta : G' \to G$ is the group monomorphism corresponding to λ.

THEOREM B.5. *Let $(B, \mathcal{A})$ be an orbifold and $\{\Omega, G, \varphi\}, \{\Omega', G', \varphi'\} \in \mathcal{A}$ two l.u.s.'s of supports $U \subseteq U'$. Let $F : \Omega \to \mathbb{C}$ be a G-invariant C^∞ function. Assume F to have compact support $\tilde{K} \subset \Omega$. Let λ be an injection of $\{\Omega, G, \varphi\}$ into $\{\Omega', G', \varphi'\}$ and $\eta : G \to G'$ the corresponding group monomorphism. Consider $\sigma'_1, \cdots, \sigma'_k \in G'$ such that $\sigma'_1 = e'$ and*

$$G'/\eta(G) = \{[\sigma'_1], \cdots, [\sigma'_k]\}$$

(and $i \neq j \Longrightarrow [\sigma'_i] \neq [\sigma'_j]$). Let $F' : \Omega' \to \mathbb{C}$ be given by

$$F'(x') = \begin{cases} 0, & x' \notin V(\Omega, \Omega') \\ F(\lambda^{-1}(\sigma_i'^{-1} x')) & x' \in \sigma'_i(\lambda(\Omega)) \end{cases} \tag{B.1}$$

where

$$V(\Omega, \Omega') = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\Omega)).$$

Then

1) F' does not depend upon the choice of representatives σ'_i of the elements in $G'/\eta(G)$.

2) F' does not depend upon the choice of injection λ.

3) F' is C^∞ differentiable, G'-invariant, and its support is contained in

$$\tilde{K}' = \bigcup_{\sigma' \in G'} \sigma'(\lambda(\tilde{K})).$$

Theorem B.5 allows us to formulate the following

DEFINITION B.6. The function $F' : \Omega \to N$ given by (B.1) is said to be the *extension* of F to $\{\Omega', G', \varphi'\}$. □

To prove 1) of Theorem B.5 let τ'_i be another representative of $[\sigma'_i]$, i.e. $\tau'_i = \sigma'_i \circ \eta(\sigma)$ for some $\sigma \in G$. Then

$$\tau'_i(\lambda(\Omega)) = \sigma'_i \circ \eta(\sigma)(\lambda(\Omega)) = \sigma'_i \lambda \sigma(\Omega) = \sigma'_i(\lambda(\Omega)).$$

Moreover, if $x' \in \tau'_i(\lambda(\Omega))$ then

$$\begin{aligned} F(\lambda^{-1}(\tau_i'^{-1}(x'))) &= F(\lambda^{-1}\eta(\sigma)^{-1}\sigma_i'^{-1}(x')) = \\ &= F(\eta(\sigma)\lambda)^{-1}\sigma_i'^{-1}(x')) = F((\lambda\sigma)^{-1}\sigma_i'^{-1}(x')) = \\ &= F(\sigma^{-1}\lambda^{-1}\sigma_i'^{-1}(x')) = F(\lambda^{-1}\sigma_i'^{-1}(x')) \end{aligned}$$

by the G-invariance of F.

The proof of 2)-3) follows from (9.4) and is left as an exercise to the reader.

THEOREM B.7. *Let $(B, \mathcal{A})$ be a C^∞ orbifold and*

$$\{\Omega, G, \varphi\}, \quad \{\Omega', G', \varphi'\} \in \mathcal{A}$$

two l.u.s.'s of supports $U \cap U' \neq \emptyset$. Assume that $U \setminus U' \neq \emptyset$ and $U' \setminus U \neq \emptyset$. Let $F' : \Omega' \to \mathbb{C}$ be a G'-invariant C^∞ function of compact support. Then there is a unique G-invariant C^∞ function $F : \Omega \to \mathbb{C}$ such that

1) $F = 0$ on $\varphi^{-1}(U \setminus U')$.

2) Given any $\{\Omega_1, G_1, \varphi_1\} \in \mathcal{A}$ of support $U_1 \subset U \cap U'$ and any injection λ of $\{\Omega_1, G_1, \varphi_1\}$ into $\{\Omega, G, \varphi\}$ we have

$$F'|_{\Omega_1} = F \circ \lambda$$

where $F'|_{\Omega_1}$ *is the restriction of* F' *to* $\{\Omega_1, G_1, \varphi_1\}$.

Theorem B.7 allows us to adopt the following

DEFINITION B.8. The function $F : \Omega \to \mathbb{C}$ (furnished by Theorem B.7) is the *prolongation* of $F' : \Omega' \to \mathbb{C}$ to $\{\Omega, G, \varphi\}$. □

To prove Theorem 9.9 let $\{\Omega_1, G_1, \varphi_1\} \in \mathcal{A}$ be a l.u.s. of support $U_1 \subseteq U \cap U'$ and consider an injection λ of $\{\Omega_1, G_1, \varphi_1\}$ into $\{\Omega . G, \varphi\}$. Let $\eta_1 : G_1 \to G$ be the group monomorphism corresponding to λ. Let $\sigma_1, \cdots, \sigma_k \in G$ represent the elements of $G/\eta_1(G_1)$ (such that $i \neq j \Longrightarrow [\sigma_i] \neq [\sigma_j]$). Let

$$F(\Omega_1, \Omega) : V(\Omega_1, \Omega) \to \mathbb{C}$$

be defined by

$$F(\Omega_1, \Omega)(x) = F_1(\lambda^{-1}(\sigma_i^{-1}(x)))$$

for any $x \in \sigma_i(\lambda(\Omega_1))$. Here $F_1 = F'|_{\Omega_1}$ is the restriction of F' to $\{\Omega_1, G_1, \varphi_1\}$ and

$$V(\Omega_1, \Omega) = \bigcup_{i=1}^{k} \sigma_i(\lambda(\Omega_1)).$$

An argument similar to that in the proof of Theorem B.5 shows that $F(\Omega_1, \Omega)$ depends neither on the choice of representatives σ_i nor on the choice of injection λ.

Let $\{\Omega_2, G_2, \varphi_2\} \in \mathcal{A}$ of support $U_2 \subseteq U_1 \subseteq U \cap U'$. Then we may consider the function $F(\Omega_2, \Omega) \to \mathbb{C}$ defined by analogy to $F(\Omega_1, \Omega)$ above. Note that $V(\Omega_2, \Omega) \subseteq V(\Omega_1, \Omega)$. We shall need the following

LEMMA B.9. $F(\Omega_1, \Omega) = F(\Omega_2, \Omega)$ *on* $V(\Omega_2, \Omega)$.

Proof. Let μ be an injection of $\{\Omega_2, G_2, \varphi_2\}$ into $\{\Omega_1, G_1, \varphi_1\}$ and $\eta_2 : G_2 \to G_1$ the corresponding group monomorphism. Let $\tau_1, \cdots, \tau_r \in G_1$ be representatives of the elements in $G_1/\eta_2(G_2)$. Then

$$\{\sigma_i \circ \eta_1(\tau_j) : 1 \leq i \leq k \,,\ 1 \leq j \leq r\}$$

represent the elements of $G/\eta_1\eta_2(G_2)$. As the definition of $F(\Omega_2, \Omega)$ does not depend upon the choice of injection (of $\{\Omega_2, G_2, \varphi_2\}$ into $\{\Omega, G, \varphi\}$) we may use the injection $\lambda \circ \mu$. We have

$$F(\Omega_2, \Omega)(x) = F_2((\lambda \circ \mu)^{-1}(\sigma_i \circ \eta_1(\tau_j))^{-1}x)$$

for any $x \in \sigma_i\eta_1(\tau_j)\lambda\mu(\Omega_2)$. Here $F_2 = F'|_{\Omega_2}$ is the restriction of F' to $\{\Omega_2, G_2, \varphi_2\}$. Yet F_2 and the restriction of F_1 to $\{\Omega_2, G_2, \varphi_2\}$ actually coincide i.e. $F_2 = F_1 \circ \mu$. Hence

$$F(\Omega_2, \Omega)(x) = F_2(\mu^{-1}\lambda^{-1}\eta_1(\tau_j)^{-1}\sigma_i^{-1}x) =$$
$$= F_1(\lambda^{-1}\eta_1(\tau_j)^{-1}\sigma_i^{-1}x) = F_1((\lambda \circ \tau_j)^{-1}\sigma_i^{-1}x)$$

for any $x \in \sigma_i\eta_1(\tau_j)\lambda\mu(\Omega_2) \subseteq \sigma_i(\lambda\mu(\Omega_2))$. The definition of $F(\Omega_1, \Omega)$ does not depend upon the choice of injection (of $\{\Omega_1, G_1, \varphi_1\}$ into $\{\Omega, G, \varphi\}$) hence we may use the injection $\lambda \circ \tau_j$. We have

$$F(\Omega_1, \Omega)(x) = F_1((\lambda \circ \tau_j)^{-1}\sigma_i^{-1}(x))$$

for any $x \in \sigma_i(\lambda(\Omega))$. Then $F(\Omega_1, \Omega) = F(\Omega_2, \Omega)$ on

$$\sigma_i\eta_1(\tau_j)\lambda\mu(\Omega_2) \subseteq \sigma_i\lambda\mu(\Omega_2) \subseteq \sigma_i\lambda(\Omega_1)$$

and Lemma B.9 is proved.

At this point we may build the function $F : \Omega \to \mathbb{C}$ aimed to in Theorem B.7. We define F to be zero on $\varphi^{-1}(U \setminus U')$. As to the set $\varphi^{-1}(U \cap U')$ we define F as follows. Let $x \in \varphi^{-1}(U \cap U')$ and set $p = \varphi(x)$. By the properties of $\mathcal{H}$ let $U_1 \in \mathcal{H}$ such that $p \in U_1 \subseteq U \cap U'$ and consider a l.u.s. $\{\Omega_1, G_1, \varphi_1\} \in \mathcal{A}$ of support U_1. Finally let us set

$$F(x) = F(\Omega_1, \Omega)(x).$$

The definition of $F(x)$ doesn't depend upon the choice of $\{\Omega_1, G_1, \varphi_1\}$ as above. Indeed let $U_2 \in \mathcal{H}$ be another open set such that $p \in U_2 \subseteq U \cap U'$ and let $\{\Omega_2, G_2, \varphi_2\} \in \mathcal{A}$ be a l.u.s. of support U_2. Then $p \in U_1 \cap U_2$ hence there is $U_3 \in \mathcal{H}$ such that $p \in U_3 \subseteq U_1 \cap U_2$. Let $\{\Omega_3, G_3, \varphi_3\} \in \mathcal{A}$ be a l.u.s. of support U_3. Then

$$F(\Omega_1, \Omega)(x) = F(\Omega_3, \Omega)(x) = F(\Omega_2, \Omega)(x)$$

by Lemma B.9 (applied to the sets $U_3 \subseteq U_1$, respectively $U_3 \subseteq U_2$ (rather than $U_2 \subseteq U_1$)). Next we show that F is G-invariant. Let $\sigma \in G$ and $x \in \Omega$. We distinguish two cases as I) $x \in \Omega \setminus \varphi^{-1}(U \cap U')$ or II) $x \in \varphi^{-1}(U \cap U')$. In the first case $\sigma(x) \in \varphi^{-1}(U \setminus U')$ and $F(\sigma(x)) = 0 = F(x)$. If the second case occurs then we may consider $\{\Omega_1, G_1, \varphi_1\} \in \mathcal{A}$ of support U_1 with $\varphi(x) \in U_1$. Let λ be an injection of $\{\Omega_1, G_1, \varphi_1\}$ into $\{\Omega, G, \varphi\}$. As $x \in \lambda(\Omega_1)$ we have $\sigma(x) \in \sigma(\lambda(\Omega_1))$ hence $F(\sigma(x)) = F_1(\lambda^{-1}(\sigma^{-1}(x))$. Yet $F(x) = F_1((\sigma \circ \lambda)^{-1}(x))$ as $x \in \lambda(\Omega)$. Therefore $F \circ \sigma = F$. To check that $F \in C^\infty$ it suffices to note that i) F is C^∞ on $\varphi^{-1}(U \cap U')$ by construction, ii) $\tilde{K} = \varphi^{-1}(\varphi'(\tilde{K}')) \subseteq \varphi^{-1}(U \cap U')$ and $\tilde{K}$ is closed in Ω, where $\tilde{K}' = \text{supp}(F') \subseteq \Omega$, iii) $F = 0$ on $\Omega \setminus \tilde{K}$, and iv) $\{\Omega \setminus \tilde{K}, \varphi^{-1}(U \cap U')\}$ is an open cover of Ω. The uniqueness of $F : \Omega \to \mathbb{C}$ (with the properties claimed in Theorem B.7) is plain.

THEOREM B.10. *Let $(B, \mathcal{A})$ be a C^∞ orbifold and $\{\Omega_0, G_0, \varphi_0\} \in \mathcal{A}$ a fixed l.u.s. of B. Let $F : \Omega_0 \to \mathbb{C}$ be a G-invariant C^∞ function of compact support. Then there is a unique C^∞ function $f : B \to \mathbb{C}$ of compact support* $\text{supp}(f) \subset U_0 = \varphi_0(\Omega_0)$ *such that $f \circ \varphi_0 = F$.*

Proof. Let $\{\Omega, G, \varphi\} \in \mathcal{A}$ be an arbitrary l.u.s. of B, of support U. We shall define a function $f_\Omega : \Omega \to \mathbb{C}$. To this end we distinguish four cases as I) $U \subseteq U_0$ or II) $U_0 \subseteq U$ or III) $U \setminus U_0 \neq \emptyset$ and $U_0 \setminus U \neq \emptyset$ and $U \cap U_0 \neq \emptyset$ or IV) $U \cap U_0 = \emptyset$. In case I, let f_Ω be the restriction of F to $\{\Omega, G, \varphi\}$. In case II, let f_Ω be the extension of F to $\{\Omega, G, \varphi\}$ (in the sense of Theorem B.5). In case III, let f_Ω be the prolongation of F to $\{\Omega, G, \varphi\}$ (in the sense of Theorem B.7). Finally, in case IV we set $f_\Omega = 0$. Let then $f : B \to \mathbb{C}$ be defined as follows. Let $p \in B$. Let $U \in \mathcal{H}$ so that $p \in U$. Let then $\{\Omega, G, \varphi\} \in \mathcal{A}$ of support U and $x \in \Omega$ so that $\varphi(x) = p$. Then set $f(p) = f_\Omega(x)$. The function f satisfies the required conditions.

THEOREM B.11. *Let B be a C^∞ orbifold and $p \in B$. For any open neighborhood U' of p there is a C^∞ map $f : B \to \mathbb{R}$ of compact support contained in U' such that $0 \leq f \leq 1$ and $f = 1$ on some compact neighborhood K of p.*

Proof. Let $\{\Omega_0, G_0, \varphi_0\}$ be a l.u.s. of B of support U_0 with $p \in U_0$. Let $x \in \Omega_0$ so that $\varphi_0(x) = p$. The set $\{\sigma(x) : \sigma \in G_0\}$ is finite. Let $\{x_1, \cdots, x_r\}$ be its elements, where $x_1 = x$ (and $r \leq n(G)$). Let $\sigma_i \in G_0$ so that $x_i = \sigma_i(x), 1 \leq i \leq r$. Let V_x' be an open neighborhood of x in Ω_0 such that $\varphi_0(V_x') \subseteq U'$ and the sets

$V'_{x_i} = \sigma_i(V'_x)$ are mutually disjoint. Let us set

$$V_x = \bigcup_{\sigma \in (G_0)_x} \sigma(V'_x).$$

Let K_0 be a compact neighborhood of x contained in V_x. Let $F : \Omega_0 \to \mathbb{R}$ be a C^∞ function with $F(\Omega_0) \subseteq [0,1]$ such that $F = 1$ on K_0 and $supp(F) \subset V_x$. Consider $f_{\Omega_0} : \Omega_0 \to \mathbb{C}$ given by

$$f_{\Omega_0} = \frac{1}{n(x)} \sum_{\sigma \in G_0} f \circ \sigma$$

where $n(x)$ is the order of $(G_0)_x$. Clearly f_{Ω_0} is a G_0-invariant C^∞ function with $f_{\Omega_0}(\Omega_0) \subseteq [0,1]$. Let us set

$$K' = \bigcap_{\sigma \in (G_0)_x} \sigma(K_0).$$

Then K' is a compact neighborhood of x. Also the restriction of f_{Ω_0} to V'_x has compact support and $f_{\Omega_0} = 1$ on K'. Let us set $K = \varphi_0(K')$. As φ_0 is continuous and open K is a compact neighborhood of p. At this point one may apply Theorem B.10. The resulting function satisfies the requirements of Theorem B.11.

A standard argument based on Theorem B.11 leads to the following

THEOREM B.12. *Let B be a C^∞ orbifold and $\{U_\alpha\}_{\alpha \in \Gamma}$ a locally finite open cover of B. There exists a countable partition of unity $\{\psi_i\}_{i \in \mathbb{N}}$ subordinate to $\{U_\alpha\}_{\alpha \in \Gamma}$ and such that each ψ_i is a C^∞ function of compact support.*

Note that the same result is stated in [**66**] only for locally finite open covers $\{U_\alpha\}_{\alpha \in \Gamma}$ whose open sets are supports of locally uniformizing systems. Also the preparatory result in Theorem B.11 is taken there (cf. the proof of Proposition 1.2 in [**66**], p. 319) for granted.

COROLLARY B.13. *Let K be a compact subset of a C^∞ orbifold B. Let U be an open subset of B such that $K \subset U$. Then there is a C^∞ function $f : B \to \mathbb{R}$ such that $f = 1$ on K and $f = 0$ on $B \setminus U$.*

Proof. Let $p \in B$. If $p \in K$ let U_p be an open neighborhood of p such that $U_p \subseteq U$. If $p \notin K$ then let U_p be an open neighborhood of p such that $U_p \cap K = \emptyset$. We produced an open cover $\{U_p\}_{p \in B}$ of B. Let $\{U_\alpha\}$ be a locally finite open refinement of $\{U_p\}_{p \in B}$. By Theorem B.12, let $\{\psi_i\}_{i \in \mathbb{N}}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. Let

$$C = \{i \in \mathbb{N} : \operatorname{supp}(\psi_i) \cap K \neq \emptyset\}.$$

Then C is a finite set and $f = \sum_{i \in C} \psi_i$ is the function aimed to in Corollary B.13.

Let X be a Banach space. By replacing $\mathbb{R}^n$ by X in the definition of the notion of an orbifold one may define the notion of a *Banach orbifold* (modelled on X). Eventually, one may request that the injections $\lambda : \Omega' \to \Omega'$ be norm preserving. Partitions of unity on Banach manifolds (modelled on a Banach space satisfying an additional smoothness condition) have been obtained by J.N. Frampton, [**109**] (by firstly constructing partitions of unity on Lindelöf spaces, cf. Theorem 1 in [**109**], p. 8). It is an open problem to construct partitions of unity on (infinitely dimensional) Banach orbifolds.

APPENDIX C

Pseudo-differential operators on $\mathbb{R}^n$

For the convenience of the reader we briefly review the main notions and results about pseudo-differential operators on $\mathbb{R}^n$ (as employed in Chapters 10 and 11). For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$D^\alpha = (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

DEFINITION C.1. The *Schwartz class* $\mathcal{S}$ is the set of all C^∞ complex valued functions f on $\mathbb{R}^n$ such that for all multi-indices α, β there is a constant $C_{\alpha,\beta} > 0$ such that

$$|x^\alpha D^\beta f| \le C_{\alpha,\beta}.$$

□

We denote by dx the measure $dx = (2\pi)^{-n/2} d\mu$ where $d\mu$ is the Lebesque measure on $\mathbb{R}^n$. As $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}$ (and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$) $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$.

DEFINITION C.2. The *convolution product* of $f, g \in \mathcal{S}$ is given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy.$$

□

The convolution product is associative and commutative. However, only approximate identities exist. Precisely, let $f \in \mathcal{S}$ with $\int f(x)dx = 1$ and set $f_u(x) = u^{-n} f(x/u), u > 0$. Then, for any $g \in \mathcal{S}$, $f_u * g$ converges uniformly to g as $u \to 0$.

DEFINITION C.3. The *Fourier transform* of $f \in \mathcal{S}$ is given by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x) dx$$

for any $\xi \in \mathbb{R}^n$. □

A standard argument based on integration by parts and Lebesgue dominated convergence theorem shows that $\hat{f} \in \mathcal{S}$ hence the Fourier transform is a map $\mathcal{S} \to \mathcal{S}$. In fact this is also bijective and the Fourier inversion formula gives its inverse (expressing f in terms of $\hat{f}$ by $f(x) = \int e^{ix\cdot\xi} \hat{f}(\xi) d\xi$). The Schwartz class $\mathcal{S}$ may be organized as a Fréchet space with the topology defined by the family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

Then $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}$ with respect to this topology. Also the Fourier transform is a homeomorphism of topological vector spaces. Both convolution and pointwise multiplication define ring structures on $\mathcal{S}$ and the Fourier transform interchanges these ring structures i.e. $\mathcal{F}(f * g) = \hat{f}\hat{g}$. Finally, the Fourier transform is an isometry with respect to the L^2 inner product

$$(f,g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$$

and, as $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$, it extends to a unitary map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $(\hat{f}, \hat{g}) = (f,g)$ (the *Plancherel theorem*).

Let $s \in \mathbb{R}$ and $f \in \mathcal{S}$. Set

$$\|f\|_s = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

DEFINITION C.4. The *Sobolev space* $H_s(\mathbb{R}^n)$ is the completion of $\mathcal{S}$ in the norm $\|\cdot\|_s$. □

Then $H_0(\mathbb{R}^n) \approx L^2(\mathbb{R}^n)$ (an isomorphism) by Plancherel theorem.

For any multi-index α, the operator D^α extends to a continuous map

$$D^\alpha : H_s(\mathbb{R}^n) \to H_{s-|\alpha|}(\mathbb{R}^n).$$

Intuitively, the number s counts the L^2 derivatives. Then one may loosely say that when extending D^α to $H_s(\mathbb{R}^n)$, $|\alpha|$ derivatives are lost.

Derivatives are also measured by means of the sup norm. Let $k \in \mathbb{Z}$, $k \geq 0$, and $f \in \mathcal{S}$. Let us set

$$\|f\|_{\infty,k} = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)|.$$

The completion of $\mathcal{S}$ in the norm $\|\cdot\|_{\infty,k}$ is contained in $C^k(\mathbb{R}^n)$.

The norms $\|\cdot\|_s$ and $\|\cdot\|_{\infty,k}$ are related as follows. Let $k \in \mathbb{Z}$, $k \geq 0$, and $s > k + n/2$. If $f \in H_s(\mathbb{R}^n)$ then f is of class C^k and

$$\|f\|_{\infty,k} \leq C\|f\|_s$$

for some $C > 0$ (the *Sobolev lemma*). This turns out to be particularly useful in showing that weak solutions (one produces for certain PDEs) are actually smooth.

Let $s > t$. Then the identity map $\mathcal{S} \to \mathcal{S}$ extends to a norm nonincreasing injection $H_s(\mathbb{R}^n) \to H_t(\mathbb{R}^n)$. This injection is compact if one restricts the supports involved. Precisely, let $f_m \in \mathcal{S}$ be a sequence of functions with supports in a fixed compact set K. Let $s > t$. If there is $C > 0$ such that $\|f_m\|_s \leq C$ for all $m \geq 1$ then there is a subsequence of $\{f_m\}$ which converges in $H_t(\mathbb{R}^n)$ (the *Rellich lemma*). The assumption that supports are uniformly bounded may not be dropped.

The space $C_0^\infty(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. Each $H_s(\mathbb{R}^n)$ is a Hilbert space hence it is isomorphic to its dual. The following invariant characterization of the dual space $H_s(\mathbb{R}^n)^*$ is also available: the L^2 pairing $\mathcal{S} \times \mathcal{S} \to \mathbb{C}$ extends to a map $H_s(\mathbb{R}^n) \times H_{-s}(\mathbb{R}^n) \to \mathbb{C}$ which identifies $H_s(\mathbb{R}^n)^*$ with $H_{-s}(\mathbb{R}^n)$.

DEFINITION C.5. A *linear partial differential operator* P *of order* m ($m \in \mathbb{Z}$, $m \geq 0$) is given by

$$P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

where $a_\alpha(x)$ are smooth. The *symbol* $p(x,\xi)$ of P is given by

$$p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha.$$

The *leading symbol* $p_L(x,\xi)$ of P is given by

$$p_L(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

□

The symbol $p(x,\xi)$ is a polynomial in ξ of degree m. The leading symbol $p_L(x,\xi)$ is a homogeneous polynomial of degree m in ξ. One of the most useful properties of the Fourier transform is that it interchanges differentiation and multiplication i.e.

$$D^\alpha f(x) = \int e^{ix\cdot\xi}\xi^\alpha \hat{f}(\xi)d\xi, \quad \xi^\alpha \hat{f}(\xi) = \int e^{-ix\cdot\xi} D^\alpha f(x)dx,$$

for any $f \in \mathcal{S}$. Consequently given a linear partial differential operator P of symbol $p(x,\xi)$ one may use the Fourier inversion formula to get

$$(Pf)(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi = \int e^{i(x-y)\cdot\xi} p(x,\xi)f(y)dyd\xi \tag{C.1}$$

for any $f \in \mathcal{S}$. It is noteworthy that the second integral is not absolutely convergent, hence one may not change the order of integration. Operators acting as in (C.1), yet corresponding to a class of symbols $p(x,\xi)$ wider than that of polynomials, are the pseudo-differential operators. Precisely we adopt the following

DEFINITION C.6. A function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a *symbol of order* $m \in \mathbb{R}$ if 1) $p(x,\xi)$ is C^∞ in (x,ξ) and has compact x-support (i.e. there is a compact set $K \subset \mathbb{R}^n$ such that $p(x,\xi) = 0$ for any $(x,\xi) \in (\mathbb{R}^n - K) \times \mathbb{R}^n$), and 2) for any multi-indices α, β there is a constant $C_{\alpha,\beta} > 0$ such that

$$\left| D_x^\alpha D_\xi^\beta p(x,\xi) \right| \le C_{\alpha,\beta}\left(1+|\xi|\right)^{m-|\beta|}.$$

□

Let S^m denote the space of all symbols of order m. If $m' \le m$ then $S^{m'} \subseteq S^m$.

DEFINITION C.7. A symbol $p(x,\xi)$ is *infinitely smoothing* if

$$p \in S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m.$$

□

DEFINITION C.8. Two symbols a, b are *equivalent* (and one writes $a \sim b$) if their difference $a - b$ is infinitely smoothing. □

DEFINITION C.9. Given a symbol $p \in S^m$ we define its associated *pseudo-differential operator* P as the linear operator $\mathcal{S} \to \mathcal{S}$ given by

$$(Pf)(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi \tag{C.2}$$

for any $f \in \mathcal{S}$. □

Then

$$\|Pf\|_{s-m} \le C\|f\|_s$$

hence P extends to a continuous map $H_s(\mathbb{R}^n) \to H_{s-m}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Therefore, if $p \in S^{-\infty}$ then $P : H_s(\mathbb{R}^n) \to H_t(\mathbb{R}^n)$ for all $s, t \in \mathbb{R}$. Finally, by the Sobolev lemma $P : H_s(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$ for all s (hence the term *infinitely smoothing* is appropriate).

DEFINITION C.10. Let $(m_j)_{j\geq 1}$ be a sequence of real numbers such that $m_j \to \infty$ as $j \to \infty$, and $p_j \in S^{m_j}$, $j \geq 1$. Given an arbitrary symbol p one writes

$$p \sim \sum_{j=1}^{\infty} p_j \tag{C.3}$$

if for any $m \in \mathbb{R}$ there is $k(m) \in \mathbb{N}$ such that

$$p - \sum_{j=1}^{k} p_j \in S^m$$

for any $k \geq k(m)$. □

The series $\sum_{j=1}^{\infty} p_j$ is not necessarily convergent and (C.3) means merely that the difference between p and the partial sums of the p_j is as smoothing as one wishes. This is the sense in which one generalizes (from differential to pseudo-differential operators) the formula expressing the symbol of a composition of two differential operators. Precisely, let P and Q be two pseudo-differential operators, of symbols $p \in S^m$ and $q \in S^{m'}$, respectively. Then PQ is a pseudo-differential operator of symbol $\sigma(PQ) \in S^{m+m'}$ satisfying

$$\sigma(PQ) \sim \sum_{\alpha} \frac{i^{\alpha}}{\alpha!}(D_{\xi}^{\alpha} p)(D_x^{\alpha} q).$$

As long as one wishes to deal with operators on compact orbifolds one may restrict the domain and range of ones operators as follows. Let $U \subseteq \mathbb{R}^n$ be an open subset with compact closure. Let $p \in S^m$ have x-support in U. We restrict the domain of the pseudo-differential operator P (associated to p) to $C_0^{\infty}(U)$ such that $P : C_0^{\infty}(U) \to C_0^{\infty}(U)$. Let $\Psi_m(U)$ denote the space of all such operators. If $m \leq m'$ then $\Psi_m(U) \subseteq \Psi_{m'}(U)$. Let us set

$$\Psi_{-\infty}(U) = \bigcap_{m \in \mathbb{R}} \Psi_m(U).$$

Let $K(x, y)$ be a C^{∞} function on $\mathbb{R}^n \times \mathbb{R}^n$ with compact x-support in U. If f is a function with compact support in U one sets

$$[P(K)f](x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy.$$

Then $P(K) \in \Psi_{-\infty}(U)$. The converse is also true. Precisely let P be a pseudo-differential operator that comes from a symbol $p(x, \xi)$ of order $-\infty$. Let $C \subset \mathbb{R}^n$ be a compact subset. Then there is a C^{∞} function $K(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, with compact support, such that $Pf = P(K)f$ for any $f \in C_0^{\infty}(\mathbb{R}^n)$ with support contained in C.

DEFINITION C.11. Let P, Q be two pseudo-differential operators. One says P, Q are *equivalent* (and one writes $P \sim Q$) if their symbols are equivalent. □

DEFINITION C.12. Let P be a differential operator and Q a pseudo-differential operator. Let $U \subset \mathbb{R}^n$ an open subset with compact closure. We say that $PQ \sim I$ *over* $C_0^{\infty}(U)$ (respectively that $QP \sim I$ *over* $C_0^{\infty}(U)$) if there exist pseudo-differential operators P', I' such that $P'Q \sim I'$ and $P'Qf = PQf$ (respectively $QP' \sim I'$ and $QP'f = QPf$) and $I'f = f$ for any $f \in C_0^{\infty}(U)$. Here I is the identity. □

DEFINITION C.13. A differential operator P is *elliptic* if its leading symbol p_L satisfies $p_L(x,\xi) = 0$ if and only if $\xi = 0$. □

We shall need the following theorem. Let P be an elliptic differential operator and $U \subseteq \mathbb{R}^n$ an open subset with compact closure. Then there is a pseudo-differential operator Q such that $PQ \sim I$ and $QP \sim I$ over $C_0^\infty(U)$.

Finally, we need to recall the invariance of pseudo-differential operators under coordinate transformations. Let $V, \tilde{V} \subset \mathbb{R}^n$ be two open sets and $f : \tilde{V} \to V$ a C^∞ diffeomorphism. Let P be a pseudo-differential operator, acting on C^∞ functions u of support contained in a compact set $K \subset V$, given by (C.2), where $p(x,\xi)$ has x-support contained in V. Let us set $\tilde{K} = f^{-1}(K)$. Given a C^∞ function Ω on $\tilde{V}$ of support contained in $\tilde{K}$, let us define $x\Omega$ by setting $(x\Omega)(\tilde{x}) = (Pu)(x)$ where $x = f(\tilde{x})$ and $u = \Omega \circ f^{-1}$. Then $x\Omega$ is a pseudo-differential operator.

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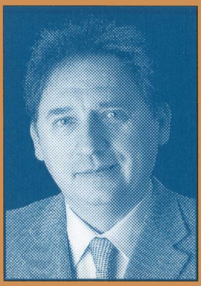
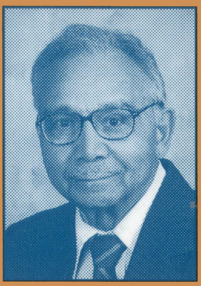

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The authors study the relationship between foliation theory and differential geometry and analysis on Cauchy–Riemann (CR) manifolds. The main objects of study are transversally and tangentially CR foliations, Levi foliations of CR manifolds, solutions of the Yang–Mills equations, tangentially Monge–Ampére foliations, the transverse Beltrami equations, and CR orbifolds. The novelty of the authors' approach consists in the overall use of the methods of foliation theory and choice of specific applications. Examples of such applications are Rea's holomorphic extension of Levi foliations, Stanton's holomorphic degeneracy, Boas and Straube's approximately commuting vector fields method for the study of global regularity of Neumann operators and Bergman projections in multi-dimensional complex analysis in several complex variables, as well as various applications to differential geometry. Many open problems proposed in the monograph may attract the mathematical community and lead to further applications of foliation theory in complex analysis and geometry of Cauchy–Riemann manifolds.

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